Modelling and Analysis for Process Control

4.1 □ INTRODUCTION

In the previous chapter, solutions to fundamental dynamic models were developed using analytical and numerical methods. The analytical integrating factor method was limited to sets of first-order linear differential equations that could be solved sequentially. In this chapter, an additional analytical method is introduced that expands the types of models that can be analyzed. The methods introduced in this chapter are tailored to the analysis of process control systems and provide the following capabilities:

1. The analytical solution of simultaneous linear differential equations with constant coefficients can be obtained using the Laplace transform method.
2. A control system can involve several processes and control calculations, which must be considered simultaneously. The overall behavior of a complex system can be modelled, considering only input and output variables, by the use of transfer functions and block diagrams.
3. The behavior of systems to sine inputs is important in understanding how the input frequency influences dynamic process performance. This behavior is most easily determined using frequency response methods.
4. A very important aspect of a system's behavior is whether it achieves a steady-state value after a step input. If it does, the system is deemed to be *stable;* if it does not, it is deemed *unstable*. Important control system analysis is based on this behavior, and the methods in this chapter are applied to determine the stability of feedback control systems in Chapter 10.

All of the methods in this chapter are limited to linear or linearized systems of ordinary differential equations. The source of the process models can be the fundamental modelling presented in Chapter 3 or the empirical modelling presented in Chapter 6.

The methods in this chapter provide alternative ways to achieve results that could, at least theoretically, be obtained for many systems using methods in Chapter 3. Therefore, the reader encountering this material for the first time might feel that the methods are redundant and unnecessarily complex. However, the methods in this chapter have been found to provide the best and *simplest* means for analyzing important characteristics of process control systems. The methods will be introduced in this chapter and applied to several important examples, but their power will become more apparent as they are used in later chapters. The reader is encouraged to master the basics here to ease the understanding of future chapters.

4.2 ▩ THE LAPLACE TRANSFORM

The Laplace transform provides the engineer with a powerful method for analyzing process control systems. It is introduced and applied for the analytical solution of differential equations in this section; in later sections (and chapters), other applications are introduced for characterizing important behavior of dynamic systems without solving the differential equations for the entire dynamic response.

The **Laplace transform** is defined as follows:

$$\mathcal{L}(f(t)) = f(s) = \int_0^\infty f(t)e^{-st}\, dt \qquad (4.1)$$

Before examples are presented, a few important properties and conventions are stated.

1. Only the behavior of the time-domain function for times equal to or greater than zero is considered. The value of the time-domain function is taken to be zero for $t < 0$.

2. A Laplace transform does not exist for all functions. Sufficient conditions for the Laplace transform to exist are (i) the function $f(t)$ is piecewise continuous and (ii) the integral in equation (4.1) has a finite value; that is, the function $f(t)$ does not increase with time faster than e^{-st} decreases with time. Functions typically encountered in the study of process control are Laplace-transformable and are not checked. Further discussion of the existence of Laplace transforms is available (Boyce and Diprima, 1986).

3. The Laplace transform converts a function in the time domain to a function in the "s-domain," in which s can take complex values. Recall that a complex number x can be expressed in Cartesian form as $A + Bj$ or in polar form as $Re^{\phi j}$ with

$$A = \text{Re}(x) \qquad B = \text{Im}(x) \qquad R = \sqrt{A^2 + B^2} \qquad \phi = \tan^{-1}\left(\frac{B}{A}\right) \qquad (4.2)$$

4. In this book, the Laplace transform of a function $T(t)$ will be designated by the argument s, as in $T(s)$. The function and its transform will be designated by the same symbol, which can be either a capital or a lowercase letter, and no overbar will be used for the transformed function. The function in the time domain will be designated as the variable (as T) or with the time shown explicitly [as $T(t)$], if needed for clarification.

5. The Laplace transform is a linear operator, because it satisfies the requirements specified in equation (3.36):

$$\mathcal{L}[aF_1(t) + bF_2(t)] = a\mathcal{L}[F_1(t)] + b\mathcal{L}[F_2(t)] \qquad (4.3)$$

6. Tables of Laplace transforms are available, so the engineer does not have to apply equation (4.1) for many commonly occurring functions. Also, these tables provide the inverse Laplace transform,

$$\mathcal{L}^{-1}[f(s)] = f(t) \qquad \text{for } t \geq 0 \qquad (4.4)$$

Since the Laplace transform is defined only for single-valued functions, the transform and its inverse are unique.

Before we proceed to the application of Laplace transforms to differential equations, equation (4.1) is applied to a few functions that will be used in later examples. A more extensive list of Laplace transforms is given in Table 4.1.

Constant

For $f(t) = C$,

$$\mathcal{L}(C) = \int_0^\infty Ce^{-st}\, dt = -\left.\frac{C}{s}e^{-st}\right|_0^\infty = \frac{C}{s} \qquad (4.5)$$

Step of Magnitude C at $t = 0$

For $\qquad f(t) = CU(t) \qquad \text{with } U(t) = \begin{cases} 0 & \text{at } t = 0^+ \\ 1 & \text{for } t > 0^+ \end{cases}$

$$\qquad (4.6)$$

$$\mathcal{L}[C(U(t))] = C\mathcal{L}[U(t)] = C\left(\int_0^\infty e^{-st}\, dt\right) = \frac{C}{s}$$

Since the variable is assumed to have a zero value for time less than zero, the Laplace transforms for the constant and step are identical.

Exponential

For $f(t) = e^{-at}$,

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at}e^{-st}\, dt = \frac{1}{a-s}\, e^{-(s-a)t}\Big|_0^\infty = \frac{1}{s-a} \qquad (4.7)$$

TABLE 4.1

Laplace transforms

No.	$f(t)$	$f(s)$
1	δ, unit impulse	1
2	$U(t)$, unit step or constant	$1/s$
3	$\dfrac{t^{n-1}}{(n-1)!}$	$1/s^n$
4	$\dfrac{1}{\tau}e^{-t/\tau}$	$\dfrac{1}{\tau s + 1}$
5	$1 + \dfrac{(a-\tau)}{\tau}e^{-t/\tau}$	$\dfrac{as+1}{s(\tau s + 1)}$
6	$\dfrac{1}{\tau^n}\dfrac{t^{n-1}e^{-t/\tau}}{(n-1)!}$	$\dfrac{1}{(\tau s + 1)^n}$
7	$\left(\dfrac{a}{\tau^2} + \dfrac{\tau - a}{\tau^3}t\right)e^{-t/\tau}$	$\dfrac{as+1}{(\tau s + 1)^2}$
8	$1 + \left(\dfrac{a-\tau}{\tau^2}t - 1\right)e^{-t/\tau}$	$\dfrac{as+1}{s(\tau s + 1)^2}$
9	$\dfrac{\tau_1 - a}{\tau_1(\tau_1 - \tau_2)}e^{-t/\tau_1} - \dfrac{\tau_2 - a}{\tau_2(\tau_1 - \tau_2)}e^{-t/\tau_2}$	$\dfrac{as+1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
10	$1 + \dfrac{\tau_1 - a}{\tau_2 - \tau_1}e^{-t/\tau_1} - \dfrac{\tau_2 - a}{\tau_2 - \tau_1}e^{-t/\tau_2}$	$\dfrac{as+1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
11	$\sin(\omega t)$	$\omega/(s^2 + \omega^2)$
12	$\cos(\omega t)$	$s/(s^2 + \omega^2)$
13	$e^{-at}\cos(\omega t)$	$\dfrac{s+a}{(s+a)^2 + \omega^2}$
14	$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2 + \omega^2}$
15	$\dfrac{C}{\tau}e^{-\xi t/\tau}\sin\left(\dfrac{\sqrt{1-\xi^2}}{\tau}t + \phi\right)$ $$C = \sqrt{\dfrac{\frac{a^2}{\tau^2} - \frac{2\xi a}{\tau} + 1}{1 - \xi^2}} \quad \phi = \tan^{-1}\left(\dfrac{\frac{a}{\tau}\sqrt{1-\xi^2}}{1 - \frac{a\xi}{\tau}}\right)$$	$\dfrac{as+1}{\tau^2 s^2 + 2\xi\tau s + 1}$
16	$\dfrac{\omega\tau e^{-t/\tau}}{1 + \omega^2\tau^2} + \dfrac{\sin(\omega t + \phi)}{\sqrt{1 + \omega^2\tau^2}}$ $\phi = \tan^{-1}(-\omega\tau)$	$\dfrac{\omega}{(\tau s + 1)(s^2 + \omega^2)}$
17	$1 - \dfrac{1}{\sqrt{1-\xi^2}}e^{-\xi t/\tau}\sin\left(\dfrac{\sqrt{1-\xi^2}}{\tau}t + \phi\right)$ $\phi = \tan^{-1}\left(\dfrac{\sqrt{1-\xi^2}}{\xi}\right)$	$\dfrac{1}{s(\tau^2 s^2 + 2\xi\tau s + 1)}$
18	$f(t) = \begin{cases} f(t-a) & t \geq a \\ 0 & t < a \end{cases}$	$e^{-as}f(s)$

a, ω, and τ_i are real and distinct, $0 < \xi \leq 1$, $n = $ integer

Sine

For $f(t) = \sin{(\omega t)}$,

$$\mathcal{L}(\sin{(\omega t)}) = \int_0^\infty \sin{(\omega t)} e^{-st}\, dt = \int_0^\infty \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) e^{-st}\, dt$$

$$= \int_0^\infty \left(\frac{e^{-(s-j\omega)t} - e^{-(s+j\omega)t}}{2j}\right) dt \qquad (4.8)$$

$$= \frac{1}{2j}\left[-\frac{e^{-(s-j\omega)t}}{(s-j\omega)} + \frac{e^{-(s+j\omega)t}}{(s+j\omega)}\right]_0^\infty = \frac{\omega}{s^2 + \omega^2}$$

Pulse

For $f(t) = C[U(0) - U(t_p)] = C/t_p$ for $t = 0$ to t_p, and $= 0$ for $t > t_p$, as graphed in Figure 4.1,

$$\mathcal{L}(f(t)) = \int_0^{t_p} \frac{C}{t_p} e^{-st}\, dt + \int_{t_p}^\infty 0 e^{-st}\, dt$$

$$= \frac{C}{t_p}\frac{(1 - e^{-st_p})}{s} \qquad (4.9)$$

An *impulse* function, which has zero width and total integral equal to C, is a special case of the pulse. Its Laplace transform can be determined by taking the limit of equation (4.9) as $t_p \to 0$ (and applying L'Hospital's rule) to give

$$Y(s)|_{t_p \to 0} = \lim_{t_p \to 0} \frac{C}{t_p}\frac{(1 - e^{-st_p})}{s}$$

$$= \lim_{t_p \to 0} \frac{-C(-se^{-st_p})}{s} = C \qquad (4.10)$$

FIGURE 4.1

Pulse function.

Derivative of a Function

To apply Laplace transforms to the solution of differential equations, the Laplace transform of derivatives must be evaluated.

$$\mathcal{L}\left(\frac{df(t)}{dt}\right) = \int_0^\infty \frac{df(t)}{dt} e^{-st}\, dt \qquad (4.11)$$

This equation can be integrated by parts to give

$$\mathcal{L}\left(\frac{df(t)}{dt}\right) = -\int_0^\infty f(t)(-s)e^{-st}\, dt + f(t)e^{-st}\Big|_0^\infty$$

$$= sf(s) - f(t)|_{t=0} \qquad (4.12)$$

The method can be extended to a derivative of any order by applying the integration by parts several times to give

$$\mathcal{L}\left(\frac{d^n f(t)}{dt^n}\right) = s^n f(s)$$

$$- \left(s^{n-1} f(t)|_{t=0} + s^{n-2}\frac{df(t)}{dt}\Big|_{t=0} + \cdots + \frac{d^{n-1} f(t)}{dt^{n-1}}\Big|_{t=0}\right) \qquad (4.13)$$

Integral

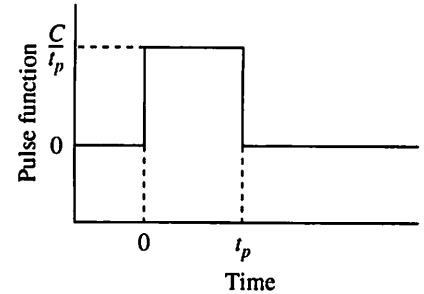By similar application of integration by parts the Laplace transform of an integral of a function can be shown to be

$$\mathcal{L}\left(\int_0^t f(t')\, dt'\right) = \int_0^\infty \left(\int_0^t f(t')\, dt'\right) e^{-st}\, dt$$

$$= \int_0^\infty \frac{e^{-st}}{s} f(t)\, dt + \left[\left(\int_0^t f(t)\, dt\right)\frac{e^{-st}}{s}\right]_{t=0}^\infty = \frac{1}{s} f(s) \tag{4.14}$$

Differential Equations

One of the main applications for Laplace transforms is in the analytical solution of ordinary differential equations. The key aspect of Laplace transforms in this application is demonstrated in equation (4.13), which shows that the transform of a derivative is an algebraic term. Thus, a differential equation is transformed into an algebraic equation, which can be easily solved using rules of algebra. The challenge is to determine the inverse Laplace transform to achieve an analytical solution in the time domain. In some cases, determining the inverse transform can be complex or impossible; however, methods shown in this section provide a general approach for many systems of interest in process control. First, the solutions of a few simple models involving differential equations, some already formulated in Chapter 3, are presented.

EXAMPLE 4.1.

The continuous stirred-tank mixing model formulated in Example 3.1 is solved here. The fundamental model in *deviation variables* is

$$V\frac{dC_A'}{dt} = F(C_{A0}' - C_A') \tag{4.15}$$
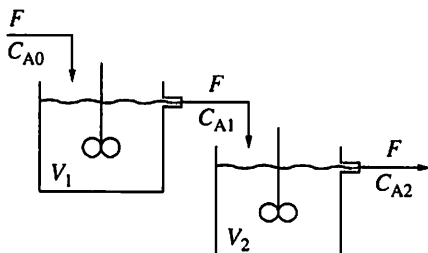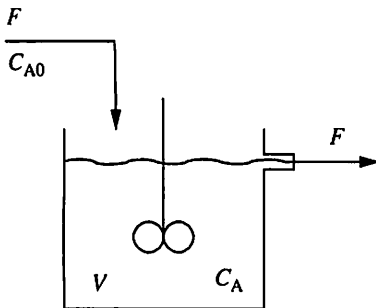
The Laplace transform is taken of each term in the model:

$$V\left[sC_A'(s) - C_A'(t)\big|_{t=0}\right] = F\left[C_{A0}'(s) - C_A'(s)\right] \tag{4.16}$$

The initial value of the tank concentration, expressed as a deviation variable, is zero, and the deviation of the inlet concentration is constant at the step value for $t > 0$; that is, $C_{A0}'(s) = \Delta C_{A0}/s$. Substituting these values and rearranging equation (4.16) gives

$$C_A'(s) = \frac{\Delta C_{A0}}{s}\frac{1}{\tau s + 1} \quad \text{with } \tau = \frac{V}{F} = 24.7\,\text{min} \tag{4.17}$$

The inverse transform of the expression in equation (4.17) can be determined from entry 5 of Table 4.1 to give the same expression as derived in Example 3.1.

$$C_A'(t) = \Delta C_{A0}(1 - e^{-t/\tau}) \tag{4.18}$$

EXAMPLE 4.2.

The model for the two chemical reactors in Example 3.3 is considered here, and the time-domain response to a step change is to be determined. The *linear* component material balances derived in Example 3.3 are repeated below in deviation variables.

$$V_1\frac{dC_{A1}'}{dt} = F(C_{A0}' - C_{A1}') - V\,kC_{A1}' \tag{3.24}$$

$$V_2 \frac{dC'_{A2}}{dt} = F(C'_{A1} - C'_{A2}) - VkC'_{A2} \qquad (3.25)$$

The Laplace transforms of the component material balances in deviation variables, noting that the initial conditions are zero, are

$$sVC'_{A1}(s) = F\left(C'_{A0}(s) - C'_{A1}(s)\right) - VkC'_{A1}(s) \qquad (4.19)$$

$$sVC'_{A2}(s) = F\left(C'_{A1}(s) - C'_{A2}(s)\right) - VkC'_{A2}(s) \qquad (4.20)$$

These equations can be combined into one equation by eliminating $C'_{A1}(s)$ from the second equation. First, solve for $C'_{A1}(s)$ in equation (4.19):

$$C'_{A1}(s) = \frac{\left(\dfrac{F}{F + Vk}\right)}{\tau s + 1} C'_{A0} \qquad (4.21)$$

This expression can be substituted into equation (4.20) along with the input step disturbance, $C'_{A0}(s) = \Delta C_{A0}/s$, to give

$$C'_{A2}(s) = \frac{K_P \Delta C_{A0}}{s(\tau s + 1)^2} \qquad (4.22)$$

with $\quad \tau = \dfrac{V}{F + VK} = 8.25\,\text{min} \qquad \Delta C_{A0} = 0.925\,\dfrac{\text{mole}}{\text{m}^3}$

$$K_P = \left(\frac{F}{F + Vk}\right)^2 = 0.448 \qquad C_{A2_s} = 0.414\,\frac{\text{mole}}{\text{m}^3}$$

The inverse transform can be determined from entry 8 of Table 4.1 to give the resulting time-domain expression for the concentration in the second reactor.

$$C'_{A2} = \Delta C'_{A0} K_P \left[1 - \left(1 + \frac{t}{\tau}\right) e^{-t/\tau}\right]$$

$$C_{A2}(t) = 0.414 + 0.414(1 - e^{-t/8.25}) - 0.050t e^{-t/8.25} \qquad (4.23)$$

This is the same result as obtained in Example 3.3.

Time Translation or Dead Time

The Laplace transform for a dead time of θ units of time is

$$\mathcal{L}[f(t - \theta)] = \int_0^\infty f(t - \theta)e^{-st}\,dt = e^{-\theta s} \int_0^\infty f(t - \theta)e^{-s(t - \theta)}\,d(t - \theta)$$
$$\qquad (4.24)$$
$$= e^{-\theta s} \int_0^\infty f(t')e^{-st'}\,dt' = e^{-\theta s} f(s)$$

When changing variables from $(t - \theta)$ to t', the lower bound of the integral remained at 0 (did not change to $t' - \theta$), because the function is defined $f(t) = 0$ for $t < 0$ for the Laplace transform. The expression in equation (4.31) is used often in process modelling to represent behavior in which the output variable does not respond immediately to a change in the input variable; this condition is often referred to as *dead time*.
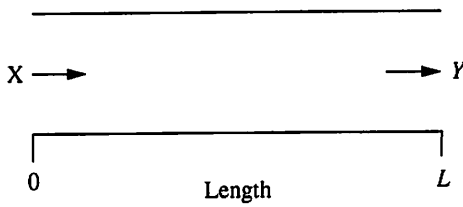
X ——▶   ——▶ Y

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0            Length            L
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FIGURE 4.2

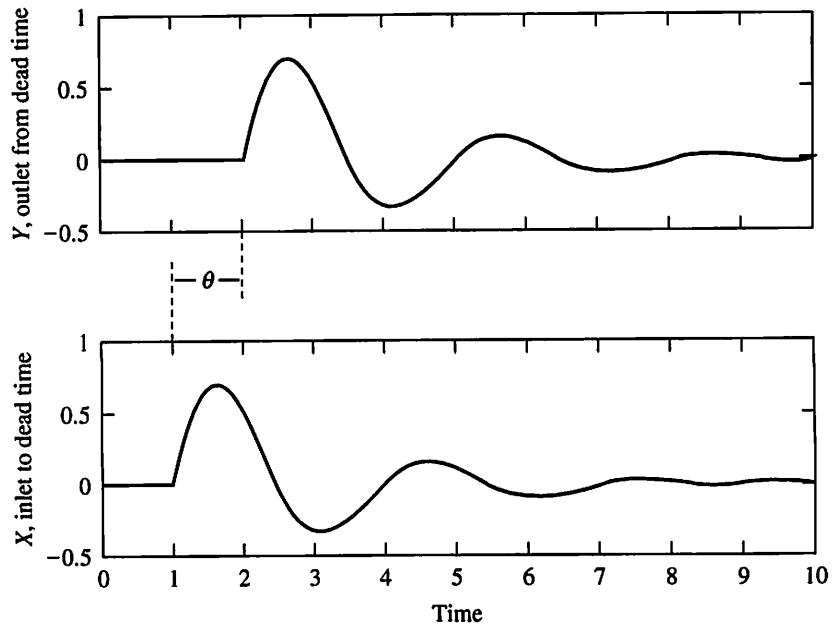**Schematic of plug flow process.**

FIGURE 4.3

Input and output for dead time (θ) of one time unit.

EXAMPLE 4.3.

The dynamic behavior of turbulent fluid flow in a pipe approximates plug flow, with the fluid properties like concentration and temperature progressing down the pipe as a front. The dead time is $\theta = L/v$, i.e., the length divided by the fluid velocity. The inlet concentration to the pipe in Figure 4.2 is X, and the outlet concentration is Y. For ideal plug flow,

$$Y(t) = X(t - \theta) \tag{4.25}$$

The Laplace transform of this model can be evaluated using the results in equation (4.24) to give

$$Y(s) = X(s)e^{-\theta s} \tag{4.26}$$

The effect of a dead time for an arbitrary input concentration is shown in Figure 4.3.

Final Value Theorem

The final condition of the transient can be determined by applying the expression for the derivative of a function, equations (4.11) and (4.12), and taking the limit as $s \to 0$.

$$\lim_{s \to 0} \left[\int_0^\infty \frac{df(t)}{dt} e^{-st} \, dt \right] = \lim_{s \to 0} [sf(s) - f(t)]_{t=0} \tag{4.27}$$

Changing the order of the limit and the integral gives

$$\int_0^\infty \frac{df(t)}{dt} \, dt = \lim_{s \to 0}[sf(s) - f(t)]\Big|_{t=0}$$

$$f(\infty) - f(t)|_{t=0} = \lim_{s \to 0}[sf(s) - f(t)]\Big|_{t=0} \tag{4.28}$$
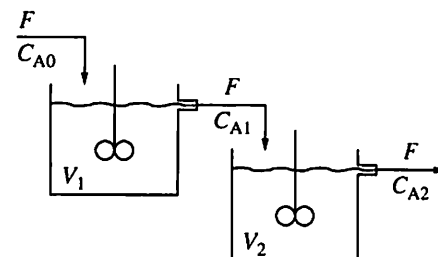
$$f(\infty) = \lim_{s \to 0} sf(s)$$

Equation (4.28) provides an easy manner for finding the final value of a variable; however, one should recognize that a simpler method would be to formulate and solve the steady-state model directly. The final value theorem finds use because the dynamic models are required for process control, and the final value can be easily determined from the Laplace transform without further modelling effort. Also, it is important to recall that the final value is exact only for a truly linear process and is approximate when based on a linearized model of a nonlinear process.

EXAMPLE 4.4.

Find the final value of the reactor concentration, expressed as a deviation from the initial value, for the CSTRs in Example 4.2. The Laplace transform for the concentration in response to a step in the inlet concentration is given in equation (4.22). The final value theorem can be applied to give

$$\lim_{s \to 0} s(C'_{A2}(s)) = \lim_{s \to 0} \left(s \frac{\Delta C_{A0}}{s} \frac{K_p}{(\tau s + 1)} \right) = K_p \Delta C_{A0} = \left(\frac{F}{F + Vk} \right)^2 \Delta C_{A0} \quad (4.29)$$

Note that this is the final value, which gives no information about the trajectory to the final value.

The engineer must recognize a limitation when applying the final value theorem. The foregoing derivation is not valid for a Laplace transform $f(s)$ that is not continuous for all values of $s \geq 0$ (Churchill, 1972). If the transform has a discontinuity for $s \geq 0$, the time function $f(t)$ does not reach a final steady-state value, as will be demonstrated in the discussion of partial fractions. Therefore, *the final value theorem cannot be applied to unstable systems.*

EXAMPLE 4.5.

Find the final value for the following system.

$$Y(s) = (\Delta X) \frac{K}{s(\tau s - 1)} \quad \text{with } \tau > 0$$

This transfer function has a discontinuity at $s = 1/\tau > 0$; therefore, the final value theorem does not apply. The analytical expression for $Y(t)$ is
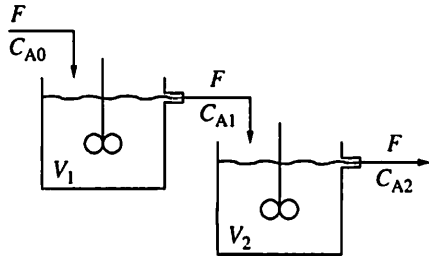
$$Y(t) = (\Delta X)K(1 - e^{t/\tau}) \quad (4.30)$$

The value of $Y(t)$ approaches negative infinity as time increases; this is not equal to the *incorrect* result from applying the final value theorem to the transfer function $-K(\Delta X)$.

Initial Value Theorem

The initial value of a variable can be determined using the initial value theorem. The derivation begins in the same manner as the final value theorem, except that the limit is taken as $s \to \infty$. Again, the order of the limit and integration is changed, resulting in the following equation:

$$\text{Initial value theorem} \qquad f(t)|_{t=0} = \lim_{s \to \infty} sf(s) \quad (4.31)$$

EXAMPLE 4.6.

The model for the two series CSTR chemical reactors in Examples 3.3 and 4.2 is considered in this example with the alteration that the volumes of the two reactors are *not the same volume*, $V_1 = 1.4$ and $V_2 = 0.70$ m^3. Determine the time-domain response.

The Laplace transforms of the two linear component material balance models in deviation variables are

$$sV_1 C'_{A1}(s) = F[C'_{A0}(s) - C'_{A1}(s)] - V_1 k C'_{A1}(s) \tag{4.32}$$

$$sV_2 C'_{A2}(s) = F[C'_{A1}(s) - C'_{A2}(s)] - V_2 k C'_{A2}(s) \tag{4.33}$$

These equations can be combined into one equation by solving for $C'_{A1}(s)$ in equation (4.32) and substituting this into equation (4.33). Also, the input step disturbance can be substituted, $C'_{A0}(s) = \Delta C_{A0}/s$, to give

$$C'_{A2}(s) = \frac{K_p C'_{A0}(s)}{(\tau_1 s + 1)(\tau_2 s + 1)} = \frac{K_p \Delta C_{A0}}{s(\tau_1 s + 1)(\tau_2 s + 1)} \tag{4.34}$$
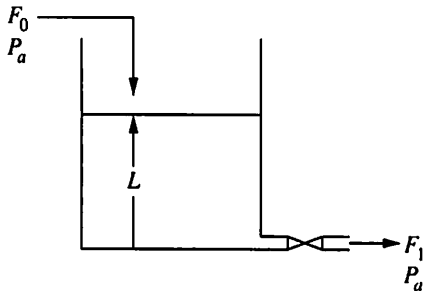
with

$$\tau_1 = \frac{V_1}{F + V_1 k} \qquad \tau_2 = \frac{V_2}{F + V_2 k} \qquad K_p = \left(\frac{F}{F + V_1 k}\right)\left(\frac{F}{F + V_2 k}\right)$$

The inverse Laplace transform can be evaluated using entry 10 in Table 4.1 (with $a = 0$) to determine the time-domain behavior of the concentration in the second reactor.

$$C'_{A2}(t) = K_p \Delta C_{A0}\left(1 + \frac{\tau_1}{\tau_2 - \tau_1}e^{-t/\tau_1} - \frac{\tau_2}{\tau_2 - \tau_1}e^{-t/\tau_2}\right) \tag{4.35}$$

This response is a smooth s-shaped curve, but it has different values at every time from the original CSTR system.

EXAMPLE 4.7.

Using Laplace transforms, determine the response of the level in the draining tank (Example 3.6) to two different changes to the inlet flow, (a) a step and (b) an impulse.

Data. Cross-section area $A = 7$ m^2, initial flows in and out, $= 100$ m^3/min, initial level $= 7$ m, $k_{F1} = 37.8$ (m^3/h)/(m$^{-0.5}$). The model for the draining tank level is based on an overall material balance of liquid in the tank depending on the flow in (F_0) and out (F_1).

$$\rho A \frac{dL}{dt} = \rho F_0 - \rho F_1 \tag{4.36}$$

The tank cross-sectional area is A. The flow out depends on the level in the tank through a nonlinear relationship, and after linearization, the level model is

$$\tau \frac{dL'}{dt} + L' = K_p F'_0 \tag{4.37}$$

with $\quad \tau = A/(0.5 k_{F1} L_s^{-0.5}) = 0.98$ h

$K_p = 1/(0.5 k_{F1} L_s^{-0.5}) = 0.14$ m/(m^3/h)

The Laplace transform of equation (4.37) can be taken to give

$$L'(s) = \frac{K_p}{\tau s + 1} F'_0(s) \tag{4.38}$$

(a) For a step change in the inlet flow rate, $F_0'(s) = \Delta F_0/(s)$; this expression can be substituted into equation (4.38), and the inverse Laplace transform can be evaluated using entry 5 in Table 4.1. The resulting expression for the draining level response to a step flow change is

$$L'(t) = K_p \Delta F_0 (1 - e^{-t/\tau}) \qquad (4.39)$$

As already determined in Example 3.6, the level dynamic response begins at its initial condition and increases in a "first-order" manner to its final value, which it reaches after about four time constants.

(b) An impulse is a change that has a finite integral but *zero duration!* Before evaluating the impulse response, we should understand how this could occur physically. For the level process, an impulse can be approximated by introducing additional liquid *very rapidly;* one method for implementing an impulse in this system would be to empty a bucket of liquid into the tank very fast. The integral of the impulse is evaluated as

$$\int F_0'(t)\, dt = M \qquad \text{m}^3 \qquad (4.40)$$

The Laplace transform of the impulse, $F_0'(s) = M$, can be introduced into equation (4.38) to give

$$L'(s) = \frac{K_p M}{\tau s + 1}$$

The inverse Laplace transform can be evaluated using entry 4 in Table 4.1, which gives [substituting the definition of the gain $K_p = \tau/(A)$] the following result:

$$L'(t) = \frac{K_p M}{\tau} e^{-t/\tau} = \frac{M}{A} e^{-t/\tau} \qquad (4.41)$$

The dynamic response of the draining tank level to an impulse of $M = 20\,\text{m}^3$ is shown in Figure 4.4. For the parameters in the example, the levels calculated using the nonlinear and linearized models are nearly identical. The level immediately increases in response to the addition of liquid. Since the inlet flow returns to its initial value after the impulse, the level slowly returns to its initial value.

Partial Fractions

The Laplace transform method for solving differential equations could be limited by the entries in Table 4.1, and with so few entries, it would seem that most models could not be solved. However, many complex Laplace transforms can be expressed as a linear combination of a few simple transforms through the use of partial fraction expansion. Once the Laplace transform can be expressed as a sum of simpler elements, each can be inverted individually using the entries in Table 4.1, thus greatly increasing the number of equations that can be solved. More importantly, the application of partial fractions provides very useful generalizations about the forms of solutions to a wide range of differential equation models, and these generalizations enable us to establish important characteristics about a system's time-domain behavior *without* determining the complete transient solution. The partial fractions method is summarized here and presented in detail in Appendix H.
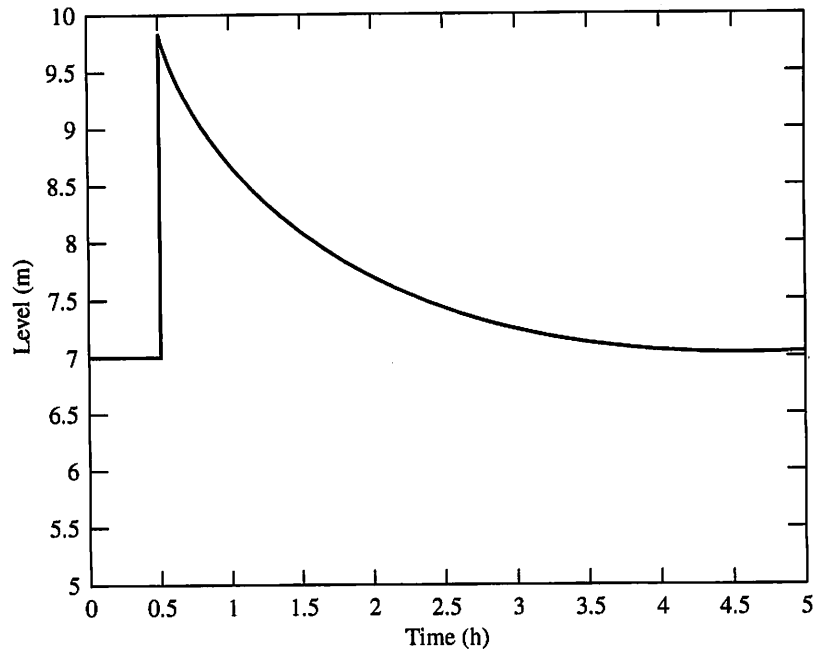
FIGURE 4.4

Response of the draining tank level in Example 4.7 to an impulse in
the flow in (F_0) at time $t = 0.50$ h.

The reader may have noticed that nearly all Laplace transforms encountered to
this point are ratios of polynomials in the Laplace variable s. The partial fractions
method can be used to express a ratio of polynomials as a sum of simpler terms.
For example, if the roots of the denominator are distinct, a ratio of higher-order
numerator and denominator polynomials can be expressed as the sum of terms, all
of which have constant numerators and first-order denominators, as given below.

$$Y(s) = N(s)/D(s) = C_1/(s - \alpha_1) + C_2/(s - \alpha_2) + \cdots \qquad (4.42)$$

with $Y(s)$ = Laplace transform of the output variable
       $N(s)$ = numerator polynomial in s of order m
       $D(s)$ = denominator polynomial in s of order n ($n > m$), termed the
               characteristic polynomial
       C_i = constants evaluated for each problem
       α_i = distinct roots of $D(s) = 0$

The inverse Laplace transform of the original term $Y(s)$, which might not appear
in a table of Laplace transforms, is the sum of the inverses of the simpler terms
$C_i/(s - \alpha_i)$, which appear as entry 4 in Table 4.1. This method is extended to
repeated and complex denominator roots in Appendix H, where it is applied to
determining the inverse of a complicated Laplace transform. However, the major
usefulness for partial fractions is in proving how several key aspects of a variable's
behavior can be determined directly from the Laplace transform *without solving
for the inverse*.

One key finding is summarized here, and another will be developed in Section
4.5 on frequency response. For any differential equation which can be arranged

into the form of equation (4.42), the inverse Laplace transform will be of the form

$$Y(t) = A_1 e^{a_1 t} + \cdots + (B_1 + B_2 t + \cdots) e^{a_p t} + \cdots + \\ [C_1 \cos(\omega t) + C_2 \sin(\omega t)] e^{a_q t} + \cdots \quad (4.43)$$

This equation includes distinct (a_1), repeated real (a_p), and complex roots (a_q), not all of which may appear in a specific solution, in which case some of the constants $(A, B,$ or $C)$ will be zero. Two important conclusions can be drawn:

1. **Stability.** The real parts of the roots of the characteristic polynomial, $D(s)$, determine the exponents $(a$'s) in the solution. These exponents determine whether the function approaches a constant value after a long time. For example, when all real parts of the roots, i.e., all $\text{Re}(\alpha_i)$, are negative, all terms on the right of equation (4.43) approach a constant value after an initial transient; a system which tends toward a constant final value is termed *stable*. If any $\text{Re}(\alpha_i)$ is greater than zero, the function $Y(t)$ will increase (or decrease) in an unbounded manner as time increases; this is termed *unstable*.
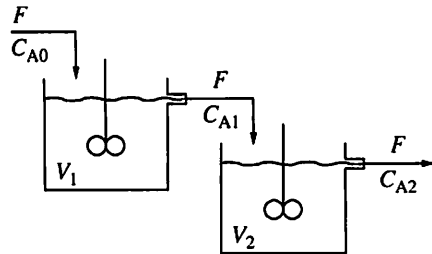
   We must look carefully at the case of roots with a value of zero. If one distinct root has a value of zero, the system is stable, while if repeated roots have values of zero, the system is unstable. This result is summarized in the following.

Number of zero roots	Terms in solution	Is the system stable?
Only one	$A_1 e^0 = A_1 =$ constant	Yes
Two (or more)	$(B_1 + B_2 t) e^0 = (B_1 + B_2 t)$	No, this term increases in magnitude without limit

2. **Damping.** The nature of the roots of the characteristic polynomial determines whether the dynamic response will experience periodic behavior for nonperiodic inputs; complex roots of $D(s)$ lead to periodic (underdamped) behavior, and real roots lead to nonperiodic (overdamped) behavior.

These two results enable the engineer to determine key features of the dynamic performance of systems without evaluating the complete dynamic transient via inverse Laplace transform. The simplification is enormous!

Certainly, a process would be easier to operate when it is stable so that variables rapidly approach constant values and no variables tend to increase or decrease without limit (based on a linearized model). Also, while oscillations are not usually completely avoided, oscillations of large magnitude are generally undesirable. Thus, the nature of the roots of the characteristic polynomial and how process design and control algorithms affect these roots are important factors in designing good processes and controls. These issues will be investigated thoroughly in Part III on feedback control by evaluating the roots of the characteristic polynomial, and the partial fraction method provides the mathematical foundation for this important analysis.

EXAMPLE 4.8.

Determine whether the concentration in the second reactor in Example 4.6 is stable and underdamped without solving for the concentration.

The roots of the denominator of the Laplace transform can be evaluated to determine these key aspects of the dynamic behavior. The Laplace transform is repeated below.

$$C'_{A2}(s) = \frac{K_p \Delta C_{A0}}{s(\tau_1 s + 1)(\tau_2 s + 1)} \qquad (4.44)$$

The roots of the denominator, which are the exponents, are $-1/\tau_1$, $-1/\tau_2$, and 0.0. Since both nonzero time constants are positive, the roots are less than zero; also, only one zero root exists. Therefore, the concentration reaches a constant value and is stable. Also, since the roots are real, the concentration is overdamped. Naturally, these conclusions are consistent with the equation defining the time-varying concentration derived in Example 4.5; however, the conclusions were reached here with minimal effort and can be determined for more complex Laplace transforms that do not appear in Table 4.1.

4.3 ☒ INPUT-OUTPUT MODELS AND TRANSFER FUNCTIONS

In some cases the values for all dependent process variables need to be determined to meet modelling goals, and the fundamental models used to this point in the book, which provide expressions for all variables, can be used in these cases. For example, the model for two series CSTRs in Example 3.3 yields expressions for the concentrations in both reactors. Some models are not unduly complex; however, detailed models can involve a large number of equations. For example, a distillation tower with 40 trays and 10 components would require over 400 differential equations.

A fundamental model solving for all dependent variables is often not required for process control, because the control system is principally involved with all input variables but only one or a few output variables. Thus, we need a method for "compressing" the model, which can be achieved by first grouping variables into three categories: input (causes), output (effects), and intermediate. For *linear* dynamic models used in process control, it is possible to eliminate intermediate variables analytically to yield an input-output model, so that intermediate variables are considered in the model even though they are not explicitly calculated. Thus, no further assumptions or simplifications are involved in input-output modelling of linear systems.

Examples of this approach have already been encountered in this chapter. For example, the basic model for two series CSTRs in Example 4.6 included equations for the concentrations in both reactors, equations (4.32) and (4.33). After the Laplace transforms are taken and the equations are combined into one equation, the model in equation (4.34) involves only the input, $C'_{A0}(s)$, and the output, $C'_{A2}(s)$. The intermediate variable, $C'_{A1}(s)$, was eliminated, although all effects of the first reactor are represented in the model.

A very common manner for presenting input-output models, which finds considerable application in process control, is the transfer function. The *trans-*

fer function is a model based on Laplace transforms with special assumptions, as follows.

The **transfer function** of a system is defined as the Laplace transform of the output variable, $Y(t)$, divided by the Laplace transform of the input variable, $X(t)$, with all initial conditions equal to zero.

$$\text{Transfer function} = G(s) = \frac{Y(s)}{X(s)} \qquad (4.45)$$

The assumptions of $Y(0) = 0$ and $X(0) = 0$ are easily achieved by expressing the variables in the transfer function as deviations from the initial conditions. Thus, all transfer functions involve variables that are expressed as deviations from an initial steady state. All derivatives are zero if the initial conditions are at steady state. (Systems having all zero initial conditions are sometimes referred to as "relaxed.") These zero initial conditions are assumed for all systems represented by transfer functions used in this book; therefore, the prime symbol "*'*" for deviation variables is redundant and is not used here when dealing with transfer functions. Transfer functions will be represented by $G(s)$, with subscripts to denote the particular input-output relationship when more than one input-output relationship exists. Before proceeding with further discussion of transfer functions, a few examples are given.

EXAMPLE 4.9.

Derive the transfer functions for the systems in Examples 4.1 and 4.2. The Laplace transform of the model in Example 4.1 is in equation (4.16). This can be rearranged to give the transfer function for this system:

$$\text{Example 4.1:} \qquad \frac{C_A(s)}{C_{A0}(s)} = \frac{1}{\tau s + 1} \qquad \text{with } \tau = \frac{V}{F} \qquad (4.46)$$

The Laplace transform for the model in Example 4.2 can be rearranged to give the transfer function for this system:

$$\text{Example 4.2:} \qquad \frac{C_{A2}(s)}{C_{A0}(s)} = \frac{K_p}{(\tau s + 1)^2} \qquad (4.47)$$

with $\qquad \tau = \dfrac{V}{F + Vk}$

$\qquad K_p = \left(\dfrac{F}{F + Vk} \right)^2$

The models from the previous examples could be used to form transfer functions, because they were in terms of deviation variables with zero initial conditions.

Note that the transfer function relates *one* output to *one* input variable. If more than one input or output exists, an individual transfer function is defined for each input-output relationship. Since the transfer function is a linear operator (as

a result of the zero initial conditions), the effects of several inputs can be summed to determine the net effect on the output.

EXAMPLE 4.10.

Derive the transfer functions for the single CSTR with the first-order reaction in Example 3.2 for changes in the inlet concentration and the feed flow rate.

The two linear models for each input change can be determined by assuming that all other inputs are constant. The basic model was derived in Example 3.2 and is repeated below.

$$V\frac{dC_A}{dt} = F(C_{A0} - C_A) - VkC_A \tag{4.48}$$

To determine whether the model is linear or not, the constant values are substituted (noting that the flow and inlet concentrations are now variables) to give

$$(2.1)\frac{dC_A}{dt} = F(C_{A0} - C_A) - (2.1)(0.040)C_A$$

The model is nonlinear because of the products of flow times concentrations. Two linearized models can be derived from equation (4.48), one for each input (assuming the other input constant), to give

$$\tau_{CA}\frac{dC_A'}{dt} + C_A' = K_{CA}C_{A0}' \quad \text{with } \tau_{CA} = \left(\frac{V}{F_s + Vk}\right) \quad K_{CA} = \left(\frac{F_s}{F_s + Vk}\right) \tag{4.49}$$

$$\tau_F\frac{dC_A'}{dt} + C_A' = K_F F' \quad \text{with } \tau_F = \left(\frac{V}{F_s + Vk}\right) \quad K_F = \left(\frac{C_{A0} - C_{As}}{F_s + Vk}\right) \tag{4.50}$$

Taking the Laplace transforms and rearranging yields the two transfer functions, one for each input.

$$\frac{C_A(s)}{C_{A0}(s)} = G_{CA}(s) = \frac{K_{CA}}{\tau_{CA}s + 1} \tag{4.51}$$

$$\frac{C_A(s)}{F(s)} = G_F(s) = \frac{K_F}{\tau_F s + 1} \tag{4.52}$$

These models and transfer functions give the behavior of the system output for individual changes in each input. If both inputs change, the overall effect is *approximately* the sum of the two individual effects. (If the process were truly linear, the total effect would be *exactly* the sum of the two individual effects.) Readers may want to return to Section 3.4 to refresh their memory on linearization.

The transfer function clearly shows some important properties of the system briefly discussed below.

Order

The *order* of the system is the highest derivative of the output variable in the defining differential equation, when expressed as a combination of all individual equations. For transfer functions of physical systems, the order can be easily determined to be the highest power of s in the denominator.

Pole

A *pole* is defined as a root of the denominator of the transfer function; thus, it is the same as a root of the characteristic polynomial. Important information on the dynamic behavior of the system can be obtained by analyzing the poles, such as

1. The stability of the system
2. The potential for periodic transients, as shown clearly in equation (4.43)

The analysis of poles is an important topic in Part III on feedback systems, since feedback control affects the poles.

Zero

A *zero* is a root of the numerator of the transfer function. Zeros do not influence the exponents $(\text{Re}(\alpha))$, but they influence the constants in equation (4.43). This can most easily be seen by considering a system with n distinct poles subject to an impulse input of unity. The expression for the output, since the Laplace transform of the unity input impulse is 1, is

$$Y(s) = G(s)X(s) = G(s) = \frac{N(s)}{D(s)} = \frac{M_i(s)}{s - \alpha_i} \quad \text{for } i = 1, n \qquad (4.53)$$

For a system with no zeros, the numerator would be equal to a constant, $N(s) = K$, and the constant associated with each root is

$$\text{With no zeros} \quad C_i = \left(\frac{K}{D_i(s)} \right)_{s = -\alpha_i} \qquad (4.54)$$

$D_i(s)$ is the denominator, with $(s - \alpha_i)$ factored out. For a system with one or more zeros, the constant associated with each root is

$$\text{With zeros} \quad C_i' = \left(K \frac{N(s)}{D_i(s)} \right)_{s = -\alpha_i} \qquad (4.55)$$

Thus, the numerator changes the weight placed on the various exponential terms. This demonstrates that the numerator of the transfer function cannot affect the stability of the system modelled by the transfer function, but it can have a strong influence on the trajectory followed by variables from their initial to final values. A simple, but less general, example to demonstrate the effect of numerator zeros is seen in the following transfer function.

$$G(s) = \frac{3s + 1}{(3s + 1)(2.5s + 1)} = \frac{1}{2.5s + 1} \qquad (4.56)$$

The numerator zero cancelled one of the poles, with the result that the second-order system behaves like a first-order system. Important examples of how zeros occur in chemical processes and how they influence dynamic behavior are presented in the next chapter, Section 5.4.

Order of Numerator and Denominator

Physical systems conform to a specific limitation between the orders of the numerator and denominator; that is, the order of the denominator must be larger than

the order of the numerator. This limitation results from the observation that real physical systems do not contain pure differentiation, as would be required for a system with a numerator order greater than the denominator order.

Causality

As discussed in Chapter 1 in the introduction of feedback control, the "direction" of the cause-effect relationship is essential to control system design. This direction is presented in the transfer function by identifying the variable in the denominator as an input (cause) and in the numerator as an output (effect). In designing feedback control strategies, the variable chosen to be adjusted must be an input, and the measured controlled variable used for determining the adjustment must be an output. When the physical system is causal, the order of the denominator is greater than that of the numerator, and the value of the transfer function as $s \to \infty$ is equal to 0. Such a transfer function is referred to as *strictly proper.*

Also, the current value of a system output variable can depend on past values of the output and inputs, but it cannot depend on future values of any variable. Therefore, the transfer function must not have prediction terms. By equation (4.24), the transfer function may not contain a term $e^{\theta s}$, which is a translation into the future (that cannot be eliminated by rearranging the transfer function). Such models are referred to as *noncausal or not physically realizable*, because they cannot represent a real physical system.

Steady-State Gain

The *steady-state gain* is the steady-state value of $\Delta Y / \Delta X$ for all systems whose outputs attain steady state after an input perturbation ΔX. The steady-state gain is normally represented by K, often with a subscript, and can be evaluated by setting $s = 0$ in the (stable) transfer function. This is exact for linear systems and gives the linearized approximation for nonlinear systems.

EXAMPLE 4.11.

Determine the stability and damping of the outlet concentration leaving the last of two isothermal CSTRs in Examples 4.2 and 4.9.

The transfer function for this system was derived in Example 4.9 and is repeated below.

$$\frac{C_{A2}(s)}{C_{A0}(s)} = G(s) = \frac{K_p}{(\tau s + 1)^2} \qquad (4.57)$$

The order of the system is the highest power of s in the denominator, 2. This indicates that the process can be modelled using two ordinary differential equations. The poles are the roots of the polynomial in the denominator; they are repeated roots, $\alpha = -1/\tau = -1/8.25 \text{ min}^{-1} = -0.1212 \text{ min}^{-1}$. The dynamic behavior is non-periodic (overdamped), because the poles are real and not complex. Also, the poles are negative, indicating that the process is stable.

For a stable process, the steady-state gain can be determined by *setting* $s = 0$ in the transfer function.

Steady-state gain: $(G(s))_{s=0} = K_p$

Also, the final value of the reactant concentration in the second reactor can be

Final value: $\displaystyle \lim_{t \to \infty} C'_{A2}(t) = \lim_{s \to 0} sC'_{A2}(s) = \lim_{s \to 0} s \left(\frac{\Delta C_{A0}}{s} \right) \frac{K_p}{(\tau s + 1)} = K_p \Delta C_{A0}$

In a specific situation the behavior of an output variable, from time 0 to completion of the response, depends on its initial conditions, input forcing, and transfer function (input-output) model. However, some very important properties of linear dynamic systems depend only on the transfer function, because the properties are independent of initial conditions and type of (bounded) forcing functions. For example, the stability of the system was shown in the previous section to be determined completely by the roots of the characteristic polynomial. The primary application of transfer functions is in the analysis of such properties of linear dynamic systems, and they are applied extensively throughout the remainder of the book.
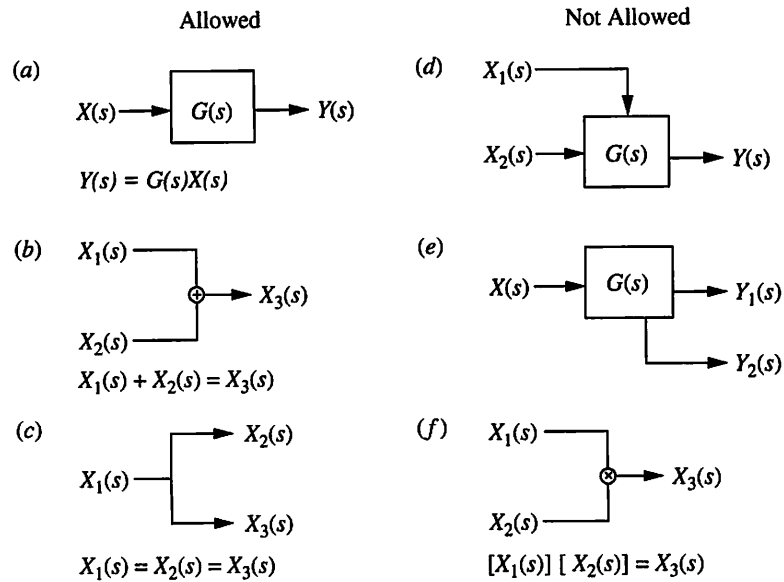
4.4 ▢ BLOCK DIAGRAMS

The transfer function introduced in the previous section describes the behavior of the individual input-output system on which it is based. Often, several different individual systems are combined, and the behavior of the combined system is to be determined. For example, a control system could involve individual systems for a reactor, a distillation tower, a sensor, a valve, and a control algorithm. The overall model could be derived by writing all equations in a large set, taking the Laplace transforms, and combining into one transfer function. Another approach retains the distinct transfer functions of the individual systems and combines these transfer functions into an overall model. This second approach is usually preferred because

1. It retains individual systems, thereby simplifying model changes (e.g., a different sensor model).
2. It provides a helpful visual representation of the cause-effect relationships in the overall system.
3. It gives insight into how different components of the system influence the overall behavior (e.g., stability).

The block diagram provides the method for combining individual transfer functions into an overall transfer function. The three allowable manipulations in a block diagram are shown in Figure 4.5a through c. The first is the transform of an input variable to an output variable using the transfer function; this is just a schematic representation of the relationship introduced in equation (4.45) and discussed in the previous section. The second is the sum (or difference) of two variables; the third is splitting a variable for use in more than one relationship. These three manipulations can be used in any sequence for combining individual models. A more comprehensive set of rules based on these three can be developed (Distephano et al., 1976), but these three are usually adequate.

To clarify, a few illegal manipulations, which are sometimes mistakenly used, are shown in Figure 4.5d through f. The first two are not allowed because the

Allowed   Not Allowed

(a)

$X(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$

$Y(s) = G(s)X(s)$

(d)

$X_1(s) \longrightarrow$

$X_2(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$

(b)

$X_1(s) \longrightarrow$

$\oplus \longrightarrow X_3(s)$

$X_2(s) \longrightarrow$

$X_1(s) + X_2(s) = X_3(s)$

(e)

$X(s) \longrightarrow \boxed{G(s)} \longrightarrow Y_1(s)$

$\longrightarrow Y_2(s)$

(c)

$\longrightarrow X_2(s)$

$X_1(s) \longrightarrow$

$\longrightarrow X_3(s)$

$X_1(s) = X_2(s) = X_3(s)$

(f)

$X_1(s) \longrightarrow$

$\otimes \longrightarrow X_3(s)$

$X_2(s) \longrightarrow$

$[X_1(s)] [X_2(s)] = X_3(s)$

FIGURE 4.5

Summary of block diagram algebra: (a–c) allowed; (d–f) not allowed.

transfer function is defined for a single input and output, and the third is not allowed because the block diagram is limited to linear operations.
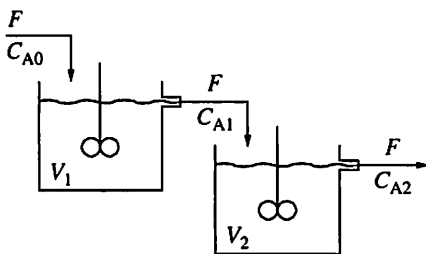
The block diagram can be prepared based on linearized models (transfer functions) of individual units and the knowledge of their interconnections. Then an input-output model can be derived through the application of block diagram algebra, which uses the three operations in Figure 4.5a through c. The model reduction steps normally followed are

1. Define the input and output variables desired for the overall transfer function.
2. Express the output variable as a function of all variables directly affecting it in the block diagram. This amounts to working in the direction opposite to the cause-effect relationships (arrows) in the diagram.
3. Eliminate intermediate variables by this procedure until only the output and one or more inputs appear in the equation. This is the input-output equation for the system.
4. If a transfer function is desired, set all but one input to zero in the equation from step 3 and solve for the output divided by the single remaining input. This step may be repeated to form a transfer function for each input.

The following examples demonstrate the principles of block diagrams, and many additional applications will be presented in later chapters.

EXAMPLE 4.12.

Draw the block diagram for the two chemical reactors in Example 4.2, and combine them into one overall block diagram and transfer function for the input C_{A0} and the output C_{A2}. The individual transfer functions are given below and shown in Figure 4.6a.
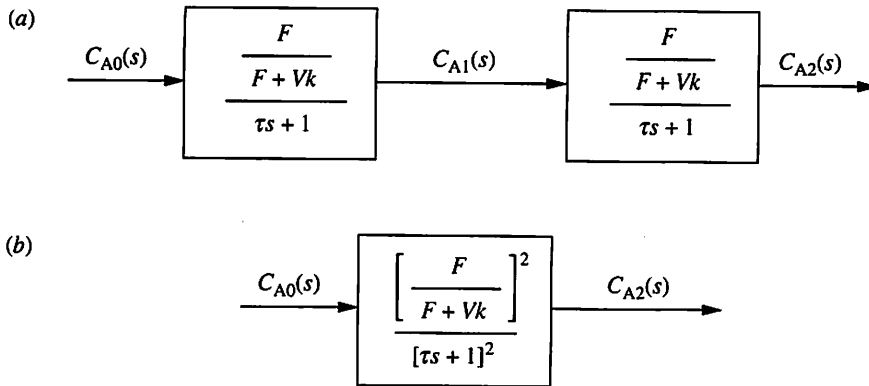
(a)

(b)

FIGURE 4.6

Block diagrams for Example 4.12.

$$G_1(s) = \frac{C_{A1}(s)}{C_{A0}(s)} = \frac{K_1}{\tau s + 1} \quad \text{with } \tau = \frac{V}{F + Vk} \quad K_1 = \frac{F}{F + Vk} \quad (4.58)$$

$$G_2(s) = \frac{C_{A2}(s)}{C_{A1}(s)} = \frac{K_2}{\tau s + 1} \quad \text{with } \tau = \frac{V}{F + Vk} \quad K_2 = \frac{F}{F + Vk} \quad (4.59)$$

Block diagram manipulations can be performed to develop the overall input-output relationship for the system.

$$C_{A2} = G_2(s)\, C_{A1}(s) = G_2(s)\, [G_1(s)\, C_{A0}(s)] = G_2(s)\, G_1(s) C_{A0}(s)$$

$$= \frac{K_1 K_2}{(\tau s + 1)^2} C_{A0}(s) \tag{4.60}$$

This can be rearranged to give the transfer function and the block diagram in Figure 4.6b.

$$\frac{C_{A2}(s)}{C_{A0}(s)} = G(s) = \frac{K_1 K_2}{(\tau s + 1)^2} \tag{4.61}$$

EXAMPLE 4.13.

Derive the overall transfer functions for the systems in Figure 4.7. The system in part (a) is a series of transfer functions, for which the overall transfer function is the product of the individual transfer functions.

$$X_n(s) = G_n(s)\, X_{n-1}(s) = G_n(s)\, G_{n-1}(s)\, X_{n-2}(s)$$

$$= G_n(s)\, G_{n-1}(s)\, G_{n-2}(s) \cdots G_1(s)\, X_0(s) \tag{4.62}$$

$$\frac{X_n(s)}{X_0(s)} = \prod_{i=1}^{n} G_i(s)$$

The system in part (b) involves a parallel structure of transfer functions, and the overall transfer function can be derived as

$$X_3(s) = X_1(s) + X_2(s) = G_1(s)\, X_0(s) + G_2(s)\, X_0(s) \tag{4.63}$$

$$\frac{X_3(s)}{X_0(s)} = G_1(s) + G_2(s)$$

(a)

(b)

(c)

FIGURE 4.7

Three common block diagram structures considered in Example 4.13.

The system in part (c) involves a recycle structure of transfer functions, and the overall transfer function can be derived as

$$X_2(s) = G_1(s) X_1(s) = G_1(s) [X_0(s) + X_3(s)] = G_1(s) [X_0(s) + G_2(s)X_2(s)]$$

$$\frac{X_2(s)}{X_0(s)} = \frac{G_1(s)}{1 - G_1(s)G_2(s)}$$

(4.64)

Examples of processes that can be represented by these structures, along with the effects of the structures on dynamic behavior, will be presented in the next chapter.

It is perhaps worth noting that the block diagram is entirely equivalent to and provides no fundamental advantage over algebraic solution of the system's linear algebraic equations (in the s domain). Either algebraic or block diagram manipulations for eliminating intermediate variables to give the input-output relationship will result in the same overall transfer function. However, as demonstrated by the examples, the block diagram manipulations are easily performed.

Two further features of block diagrams militate for their extensive use. The first is the helpful visual representation of the integrated system provided by the block diagram. For example, the block diagram in Figure 4.7c clearly indicates a recycle in the system, a characteristic that might be overlooked when working with a set of equations. The second feature of the diagrams is the clear representation of the cause-effect relationship. The arrows present the direction of these relationships and enable the engineer to identify the input variables that influence the

output variables. As a result, block diagrams are widely used and will be applied extensively in the remainder of this book.

4.5 ▦ FREQUENCY RESPONSE*

An important aspect of process (and control system) dynamic behavior is the response to periodic input changes, most often disturbances. The range of possible dynamic behavior can be determined by considering cases (in thought experiments) at different input frequencies for an example system, such as the mixing tank in Figure 4.8. If an input variation is slow, with a period of once per year, the output response would be essentially at its steady-state value (the same as the input), with the transient response being insignificant. If the input changed very rapidly, say every nanosecond, the output would not be significantly influenced; that is, its output amplitude would be insignificant. Finally, if the input varies at some *intermediate frequency* near the response time of the process, the output will fluctuate continuously at values significantly different from its mean value. The behavior at extreme frequencies is easily determined in this thought experiment, but the method for determining the system behavior at intermediate frequencies is not obvious and is useful for the design process equipment, selection of operating conditions, and formulation of control algorithms to give desired performance. Before presenting a simplified method for evaluating the effects of frequency, a process equipment design example is solved by determining the complete transient response to a periodic input.

FIGURE 4.8

Intermediate inventory to attenuate variation.

EXAMPLE 4.14.

The feed composition to a reactor varies with an amplitude larger than acceptable for the reactor. It is not possible to alter the upstream process to reduce the oscillation in the feed; therefore, a drum is located before the reactor to reduce the feed composition variation, as shown in Figure 4.8. What is the minimum volume of the tank required to maintain the variation at the inlet to the reactor (outlet of the tank) less than or equal to ± 20 g/m^3?

Assumptions. The assumptions include a constant well-mixed volume of liquid in the tank, constant density, constant flow rate in, and the input variation in concentration is well represented by a sine. Also, the system is initially at steady state.

Data.

1. $F = 1$ m^3/min.

2. C_{A0} is a sine with amplitude of 200 g/m^3 and period of 5 minutes about an average value of 200 g/m^3.

Solution. The model for this stirred-tank mixer was derived in Example 3.1 and applied in several subsequent examples. The difference in this example is that the input concentration is characterized as a sine rather than a step, $C'_{A0} = A \sin (\omega t)$. Thus, the model for the tank is

$$V \frac{dC'_{A1}}{dt} = F(A \sin (\omega t)) - FC'_{A1} \qquad (4.65)$$

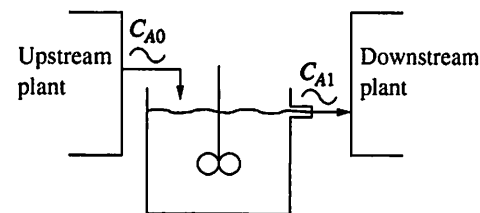To more clearly evaluate the model for linearity, the values for all constants (in this

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*The reader may choose to cover this material when reading Chapter 10.

example) can be substituted into equation (4.65), giving the following:

$$V\frac{dC'_{A1}}{dt} = (1)[(200)\sin(2\pi/5)] - (1)C'_{A1}$$

Since V is a *constant* to be determined, the equation is linear, and we can proceed without linearization. Equation (4.65) could be solved by using either the integrating factor or Laplace transforms. Here, the Laplace transform of equation (4.65) is taken to give, after some rearrangement,

$$C'_{A1}(s) = \frac{A\omega}{\tau}\frac{1}{\left(s + \frac{1}{\tau}\right)(s^2 + \omega^2)} \quad \text{with } \tau = \frac{V}{F} \tag{4.66}$$

The dynamic behavior of the concentration can be determined by evaluating the inverse of the Laplace transform. This expression appears as entry 16 in Table 4.1. The resulting expression for the time behavior is given in the following equation:

$$C'_{A1}(t) = \frac{A\omega\tau}{1 + \tau^2\omega^2}e^{-t/\tau} + \frac{A}{\sqrt{1 + \tau^2\omega^2}}\sin(\omega t + \phi) \tag{4.67}$$

Results analysis. The first term in equation (4.67) tends to zero as time increases; thus, the response of the process after a long time of operation (about four time constants) is not affected by this term. The second term describes the "long-time" behavior of the concentration in response to a sine input. It is periodic, with the *same frequency* as the input forcing and an amplitude that depends on the input amplitude and frequency, as well as process design parameters. For this example, the output amplitude must be less than or equal to 20; by setting the amplitude equal to the limit, the time constant, and thus the volume, can be calculated.

$$|C_{A1}|_{max} = \frac{A}{\sqrt{1 + \tau^2\omega^2}} = 20 \tag{4.68}$$

$$V = \tau F = F\frac{\sqrt{\left(\frac{A}{|C_{A1}|_{max}}\right)^2 - 1}}{\omega} = 1.0\frac{\sqrt{\left(\frac{200}{20}\right)^2 - 1}}{2\pi/5} = 7.9 \text{ m}^3 \tag{4.69}$$

Note that the analytical solution provides valuable sensitivity information, such as the amount the size of the vessel must be increased if the input frequency decreases.

For general frequency response analysis, periodic inputs will be limited to sine inputs, which will be a mathematically manageable problem. Also, only the "long-time" response (i.e., after the initial transient, when the output is periodic) is considered. The periodic behavior after a long time is sometimes referred to as "steady-state"; however, it seems best to restrict the term *steady-state* to describe systems with zero time derivatives.

The periodic behavior of the input and output after a long time—the frequency response—is shown in Figure 4.9, and frequency response is defined as follows:

The **frequency response** defines the output behavior of a system to a sine input after a long enough time that the output is periodic. The output (Y') of a *linear* system will be a sine with the same frequency as the input (X'), and the relationship between input and output can be characterized by

$$\text{Amplitude ratio} = \frac{\text{output magnitude}}{\text{input magnitude}} = \frac{|Y'(t)|_{max}}{|X'(t)|_{max}} \qquad (4.70)$$

Phase angle = phase difference between the input and output

For the system in Figure 4.9 the amplitude ratio = B/A, and the phase angle = $-2\pi(P'/P)$ radians. Note that P' is the time difference between the input and its effect at the output and can be greater than P.

The usefulness of the amplitude ratio was demonstrated in Example 4.14, and the importance of the phase angle, while not apparent yet, will be shown to be very important in the analysis of feedback systems. Recalling that feedback systems adjust an input based on the behavior of an output, it is reasonable that the time (or phase) delay between these variables would affect the feedback system. The analysis of feedback systems using frequency response methods is introduced in Chapter 10 and used in many subsequent chapters.

Example 4.14 demonstrates that the frequency response of linear systems can be determined by the direct solution of the ordinary differential equations. However, this approach is time-consuming for complex systems. Also, the solution of the entire transient response provides information not needed, because only the behavior after the initial transient is desired. Now a simpler approach for
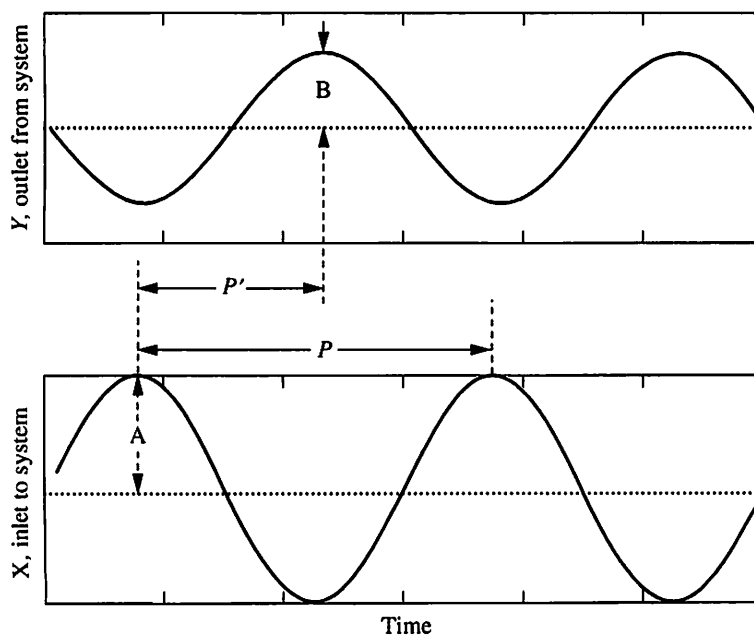

FIGURE 4.9

Frequency response for a linear system.

determining frequency response is presented; it is based on the transfer function of the system.

The following expressions, which are derived in Appendix H, show how the long-time frequency response of a linear system can be evaluated easily using the transfer function and algebraic manipulations. The long-time output Y_{FR} depends on the dynamic system model, $G(s)$, and the input sine amplitude, A, and frequency, ω.

$$Y_{FR}(t) = A\,|G(j\omega)|\sin(\omega t + \phi) \qquad (4.71)$$

The two key parameters of the frequency response can be determined from

$$\text{Amplitude ratio} = AR = \frac{|Y_{FR}|_{max}}{A} = \frac{A\,|G(j\omega)|}{A} = |G(j\omega)|$$

$$= \sqrt{\text{Re}[G(j\omega)]^2 + \text{Im}[G(j\omega)]^2} \qquad (4.72)$$

$$\text{Phase angle} = \phi = \angle G(j\omega) = \tan^{-1}\left(\frac{\text{Im}[G(j\omega)]}{\text{Re}[G(j\omega)]}\right) \qquad (4.73)$$

It is important to recognize that the frequency (ω) must be expressed as radians/time.

Thus, the frequency response can be determined by substituting $j\omega$ for s in the transfer function and evaluating the magnitude and angle of the resulting complex number! This is *significantly simpler* than solving the differential equation.

Note that the frequency response is entirely determined by the transfer function. This is logical because the initial conditions do not influence the long-time behavior of the system. Also, the derivation of the equations (4.72) and (4.73) clearly indicates that they are appropriate only for *stable systems*. If the system were unstable (i.e., if $\text{Re}(\alpha_i) > 0$ for any i), the output would increase without limit (for the linear approximation). Also, this analysis demonstrates that the output of a linear system forced with a sine approaches a sine after a sufficiently long time. How "long" this time is depends on all other terms; for most of the transient to have died out (i.e., $e^{-\alpha t} < 0.02$), the time should satisfy $\alpha t = t/\tau > 4$. Thus, a long time can usually be taken to be about four times the longest time constant, or the smallest α.

EXAMPLE 4.15.

Repeat the frequency response calculations for the mixing tank in Example 4.14 and Figure 4.8, this time using the direct method based on the transfer function. The frequency response is determined by substituting $j\omega$ for s in the first-order transfer function with $\tau = 7.9$.

$$G(s) = \frac{1}{\tau s + 1}$$

$$G(j\omega) = \frac{1}{\tau\omega j + 1} = \frac{1}{\tau\omega j + 1}\frac{1 - \tau\omega j}{1 - \tau\omega j} = \frac{1 - \tau\omega j}{1 + \tau^2\omega^2} \qquad (4.74)$$

$$AR = |G(j\omega)| = \frac{\sqrt{1 + \tau^2\omega^2}}{1 + \tau^2\omega^2} = \frac{1}{\sqrt{1 + \tau^2\omega^2}} = \frac{1}{\sqrt{1 + 62.4\omega^2}}$$

$$\phi = \angle G(j\omega) = \tan^{-1}(-\omega\tau) = \tan^{-1}(-7.9\omega) \qquad (4.75)$$

FIGURE 4.10

Frequency response for Example 4.15, $C_A(j\omega)/C_{A0}(j\omega)$, presented as a Bode plot.

A frequency response is often presented in the form of a *Bode plot*, in which the log of the amplitude and the phase angle are plotted against the log of the frequency. An example of the Bode plot for the system in Example 4.15 is given in Figure 4.10. From this result, it can be determined that the amplitude ratio is nearly equal to the steady-state gain for all frequencies below about 0.10 rad/min for this example, and it decreases rapidly as frequency increases from this value. Also, the amplitude ratio at a frequency of $2\pi/5 = 1.26$ rad/min is the desired value of 0.10. Finally, this graph clearly indicates the sensitivity of the result to potential errors in time constant and frequency; for example, the output amplitude is insensitive to frequency at low frequencies and quite sensitive at high frequencies.

EXAMPLE 4.16.

The two isothermal series CSTRs in Examples 3.3 and 4.2 rely on upstream processes for the feed of reactant A. The upstream process producing A does not

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operate exactly at steady state. Based on an analysis of the data, the feed concentration to the first reactor varies around its nominal value in a manner that can be approximated by a sine with an amplitude of 0.10 mol/m³ and frequency of 0.20 rad/min. Would the second reactor concentration deviate from its steady-state value by more than 0.05 mol/m³? Variation greater than this amount is not acceptable to the customer.

To answer this question, the frequency response must be evaluated. The important behaviors can be stated as

$$C_{A0}(t) = C_{A0ss} + A\sin(\omega t) \quad \text{or} \quad C'_{A0}(t) = A\sin(\omega t)$$

$$C_{A2}(t) = C_{A2ss} + B\sin(\omega t + \phi) \quad \text{or} \quad C'_{A2}(t) = B\sin(\omega t + \phi)$$

with $A = 0.10$ mole/m³, $\omega = 0.20$ rad/min, and B the unknown amplitude to be evaluated and compared with its maximum allowable variation magnitude. The transfer function based on component material balances for the two tanks was derived in Example 4.9 and is repeated below.

$$\frac{C_{A2}(s)}{C_{A0}(s)} = \frac{K_p}{(\tau s + 1)^2} \tag{4.76}$$

The gain is 0.448, and the time constants are both 8.25 minutes. The results in equations (4.71) to (4.73) demonstrate that the amplitude of the output variable can be evaluated by setting $s = j\omega$ and evaluating the magnitude. The expressions for the frequency responses for many common transfer functions are provided later in the book (e.g., Table 10.2), so the results of the algebraic manipulations are summarized here without intermediate derivations being shown.

$$|G(j\omega)| = \frac{K_p}{(1 + \omega^2\tau^2)} = \frac{B}{A} \tag{4.77}$$

The magnitude of the output concentration is the product of the input magnitude and the amplitude ratio. Therefore,

$$B = \frac{K_p}{(1 + \omega^2\tau^2)}A$$

$$= (0.12)(0.10 \text{ mol/m}^3) = 0.012 \text{ mol/m}^3 < 0.050 \text{ mol/m}^3 \tag{4.78}$$

Since the outlet concentration magnitude is lower than the maximum allowed, the operation would be considered acceptable, but good engineering would call for continued efforts to reduce all variation in product quality. Note that in this case, no control correction is required. We are seldom so fortunate, and we usually have to introduce corrective control actions through process control to maintain consistent product quality.

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The algebraic manipulations required to evaluate the amplitude ratio and phase angle can be tedious. However, relationships to ease hand calculations are provided in Chapter 10 for the commonly occurring series combinations of individual units. For more complex structures the frequency response can be easily evaluated using computer technology, because the amplitude ratio is the magnitude of the properly defined function of a complex variable; likewise, the phase angle is the *argument* of a complex variable. Many programming languages provide standard evaluations of these functions.

In conclusion, the frequency response of a linear system can be easily determined from the transfer function using equations (4.72) and (4.73). The frequency

response gives useful information concerning how the process behaves for various input frequencies, and these results can be used for determining equipment design parameters, such as the size of a drum to attenuate fluctuations. The general frequency responses for some common systems are given in the next chapter for several common systems, such as first- and second-order, and important applications of frequency response to the analysis of feedback control systems are covered in Part III.

4.6 ☐ CONCLUSIONS

The methods in Chapters 3 and 4 can be combined in an approach, shown in Figure 4.11, designed to provide models in the format most useful for the analysis of process control systems. The initial steps involve the modelling procedure based on fundamental principles summarized in Table 3.1. This procedure can be applied to each process in a complex plant. Then the transfer function of each system is determined by taking the Laplace transform of the linearized model. The block diagram can be constructed to present the interactions among the individual transfer functions, and the overall transfer function for the integrated system can be derived through block diagram manipulation.

The overall transfer functions can be used to determine some important properties of the system *without solving the defining differential equations*. These properties include

1. The final value of the output variable
2. The stability of the response
3. The response of the output to a sine input

Determining this information without the entire dynamic response has two advantages:

1. It reduces the effort to establish these system properties.
2. It assists in understanding the ways in which equipment design, operating conditions, and control systems affect these properties.

Naturally, information about the entire transient is not obtained by analyzing the poles of the transfer function or by the frequency response calculations. The complete transient response can be obtained if needed from analytical or numerical solution of the algebraic and differential equations.

As noted in the previous chapter, many different processes—heat exchangers, reactors, and so forth—behave in similar ways. The transfer function method presented in this chapter gives us a useful way to compare models for processes and recognize similarities and differences, which is the topic of the next chapter.

REFERENCES

Boyce, W., and R. Diprima, *Elementary Differential Equations*, Wiley, New York, 1986.

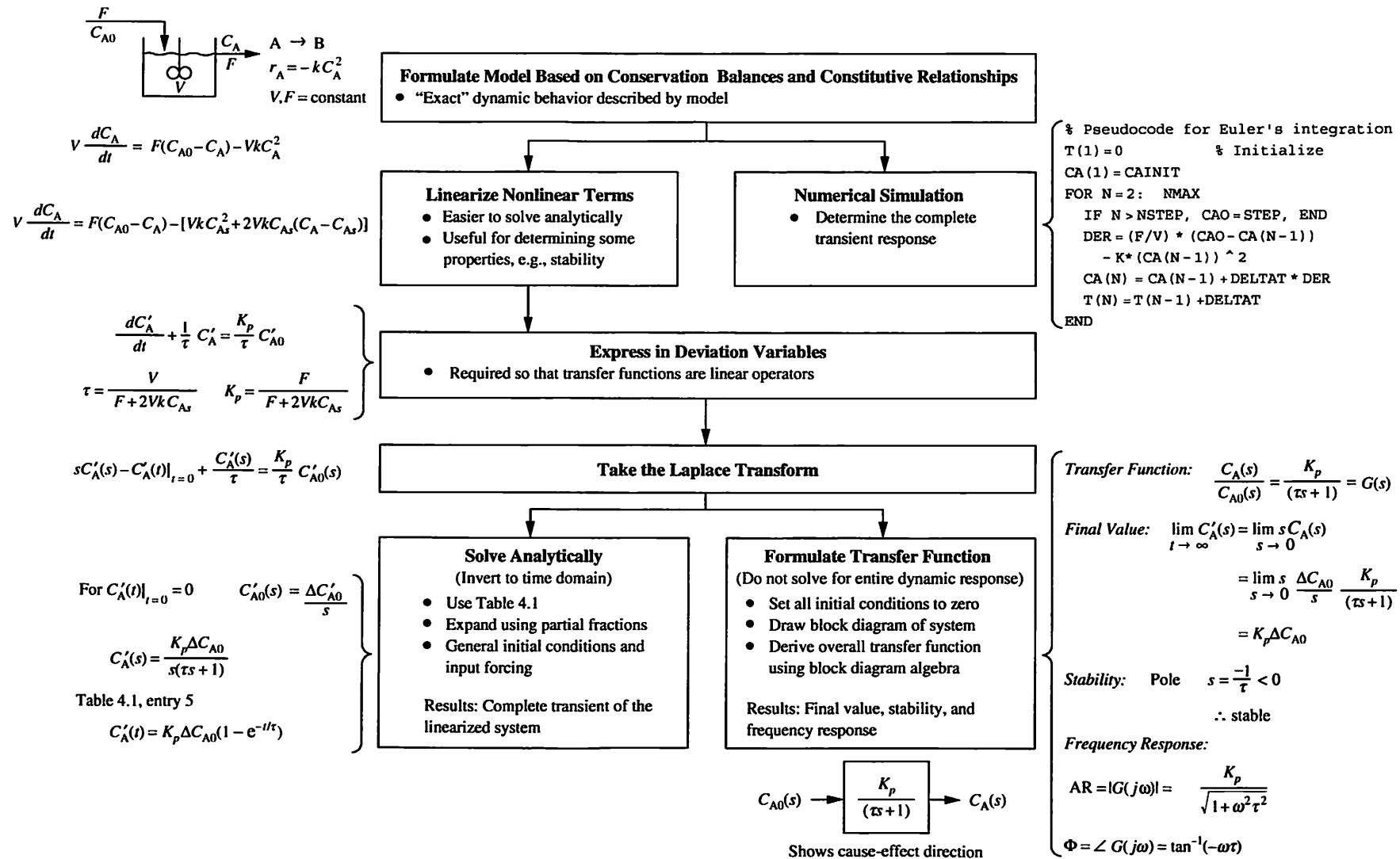Caldwell, W., G. Coon, and L. Zoss, *Frequency Response for Process Control*, McGraw-Hill, New York, 1959.

$$A \rightarrow B$$
$$r_A = -kC_A^2$$
$$V, F = \text{constant}$$

$$V\frac{dC_A}{dt} = F(C_{A0} - C_A) - VkC_A^2$$

$$V\frac{dC_A}{dt} = F(C_{A0} - C_A) - [VkC_{As}^2 + 2VkC_{As}(C_A - C_{As})]$$

$$\frac{dC_A'}{dt} + \frac{1}{\tau}C_A' = \frac{K_p}{\tau}C_{A0}'$$

$$\tau = \frac{V}{F + 2VkC_{As}} \qquad K_p = \frac{F}{F + 2VkC_{As}}$$

$$sC_A'(s) - C_A'(t)|_{t=0} + \frac{C_A'(s)}{\tau} = \frac{K_p}{\tau}C_{A0}'(s)$$

For $C_A'(t)|_{t=0} = 0 \qquad C_{A0}'(s) = \frac{\Delta C_{A0}'}{s}$

$$C_A'(s) = \frac{K_p \Delta C_{A0}}{s(\tau s + 1)}$$

Table 4.1, entry 5

$$C_A'(t) = K_p \Delta C_{A0}(1 - e^{-t/\tau})$$

Formulate Model Based on Conservation Balances and Constitutive Relationships
- "Exact" dynamic behavior described by model

Linearize Nonlinear Terms
- Easier to solve analytically
- Useful for determining some properties, e.g., stability

Numerical Simulation
- Determine the complete transient response

Express in Deviation Variables
- Required so that transfer functions are linear operators

Take the Laplace Transform

Solve Analytically
(Invert to time domain)
- Use Table 4.1
- Expand using partial fractions
- General initial conditions and input forcing

Results: Complete transient of the linearized system

Formulate Transfer Function
(Do not solve for entire dynamic response)
- Set all initial conditions to zero
- Draw block diagram of system
- Derive overall transfer function using block diagram algebra

Results: Final value, stability, and frequency response

```
% Pseudocode for Euler's integration
T(1) = 0              % Initialize
CA(1) = CAINIT
FOR N = 2:  NMAX
  IF N > NSTEP, CAO = STEP, END
  DER = (F/V) * (CAO - CA(N-1))
      - K*(CA(N-1))^2
  CA(N) = CA(N-1) + DELTAT * DER
  T(N) = T(N-1) + DELTAT
END
```

Transfer Function: $\dfrac{C_A(s)}{C_{A0}(s)} = \dfrac{K_p}{(\tau s + 1)} = G(s)$

Final Value: $\lim\limits_{t \to \infty} C_A'(s) = \lim\limits_{s \to 0} sC_A(s)$

$$= \lim\limits_{s \to 0} s \frac{\Delta C_{A0}}{s} \frac{K_p}{(\tau s + 1)}$$

$$= K_p \Delta C_{A0}$$

Stability: Pole $\quad s = \dfrac{-1}{\tau} < 0$

$$\therefore \text{ stable}$$

Frequency Response:

$$AR = |G(j\omega)| = \frac{K_p}{\sqrt{1 + \omega^2 \tau^2}}$$

$$\Phi = \angle G(j\omega) = \tan^{-1}(-\omega\tau)$$

$$C_{A0}(s) \rightarrow \boxed{\frac{K_p}{(\tau s + 1)}} \rightarrow C_A(s)$$

Shows cause-effect direction

FIGURE 4.11

Steps in developing models for process control with sample results for a chemical reactor.

Churchill, R., *Operational Mathematics*, McGraw-Hill, New York, 1972.

Distephano, S., A. Stubbard, and I. Williams, *Feedback Control Systems*, McGraw-Hill, New York, 1976.

Jensen, V., and G. Jeffreys, *Mathematical Methods in Chemical Engineering*, Academic Press, London, 1963.

Ogata, K., *Modern Control Engineering*, Prentice-Hall, Englewood Cliffs, NJ, 1990.

ADDITIONAL RESOURCES

The following references provide background on Laplace transforms and provide extensive tables.

Doetsch, G., *Introduction to the Theory and Application of Laplace Transforms*, Springer Verlag, New York, 1974.

Nixon, F., *Handbook of Laplace Transforms* (2nd ed.), Prentice-Hall, Englewood Cliffs, NJ, 1965.

Spiegel, M., *Theory and Problems of Laplace Transforms*, McGraw-Hill, New York, 1965.

Frequency responses can be determined experimentally, although at the cost of considerable disturbance to the process. This was done to ensure the concepts applied to chemical processes, as discussed in the references below, but the practice has been discontinued.

Harriott, P., *Process Control*, McGraw-Hill, New York, 1964.

Oldenburger, R. (ed.), *Frequency Response*, Macmillan, New York, 1956.

For additional discussions on the solution of dynamic problems for other types of physical systems, see Ogata, 1990 (in the References) and

Ogata, K., *System Dynamics* (2nd ed.), Wiley, New York, 1992.

Tyner, M., and F. May, *Process Control Engineering*, The Ronald Press, New York, 1968.

All of the questions in Chapter 3 relating to dynamics can be solved using methods in this chapter; thus, returning to those questions provides additional exercises. Also, when solving the questions in this chapter, it is recommended that the results be analyzed to determine

- The order of the system
- Whether the system can experience periodicity and/or instability
- The block diagram with arrows properly representing the causal relationships
- The final value

QUESTIONS

4.1. Several of the example systems considered in this chapter are analyzed concerning the violation of safety limits. A potential strategy for a safety

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system would be to monitor the value of the critical variable and when the variable approaches the safety limit (i.e., it exceeds a preset "action" value), a response is implemented to ensure safe operation. Three responses are proposed in this question to prevent the critical variable from exceeding a maximum-value safety limit, and it is proposed that each could be initiated when the measured variable reaches the action value. Critically evaluate each of the proposals, and if the proposal is appropriate, state the value of the action limit compared to the safety limit.

The proposed responses are

(i) Set the concentration in the feed (C_{A0}) to *zero*.
(ii) Set the inlet flow to zero.
(iii) Introduce an inhibitor that stops the chemical reaction (for *b*).

The critical variables and systems are

(*a*) C_A in the mixer in Example 4.1
(*b*) C_{A2} in the series of two chemical reactors in Example 4.2

4.2. Solve the following models for the time-domain values of the dependent variables using Laplace transforms.

(*a*) Example 3.2
(*b*) Example 3.2 with an impulse input and with a ramp input, $C'_{A0}(t) = at$ for $t > 0$ (with a an arbitrary constant)
(*c*) Example 3.3 with an impulse input

4.3. The room heating Example 3.4 is to be reconsidered. In this question, a mass of material is present in the room and exchanges heat with the air according to the equation $Q = UA_m(T - T_m)$, in which UA_m is an overall heat transfer coefficient between the mass and the room air, and T_m is the uniform temperature of the mass.

(*a*) Derive models for the temperatures of the air in the room and the mass. Combine them into one differential equation describing T.
(*b*) Explain how this system would behave with an on-off control and note differences, if any, with the result in Example 3.4.

4.4. An impulse of a component could be introduced into a continuous-flow mixing tank.

(*a*) Describe how the experiment could be performed; specifically, how could the impulse be implemented in the experiment?
(*b*) Derive a model for the component concentration in the tank, and solve for the concentration of the component in the tank after the impulse.
(*c*) Discuss useful information that could be determined from this experiment.

4.5. A CSTR has constant volume and temperature and is well mixed. The reaction A → B is first-order and irreversible. The feed can contain an impurity which serves as an inhibitor to the reaction; the rate of reaction of A is $r_A = -k_0 e^{-E/RT} C_A/(1 + k_1 C_I)$ where C_I is the concentration of inhibitor. The reactor is initially at steady state and experiences a step change in the inhibitor concentration. Determine the dynamic response of the concentration of reactant A after the step based on a linearized model. (Hint: You must determine the concentration of inhibitor first.)

4.6. For the following systems, (*a*) apply the final value theorem and (*b*) calculate the frequency response.

    (i) Example 3.2

    (ii) Example 3.3

    (iii) A level system with $L(s)/F_{in}(s) = 1/As$, with $F_{in}(s) = \Delta F_{in}/s$ and A = cross-sectional area [see equation (5.15)].

For each case, state whether the result is correct, and if not, why.

4.7. The process shown in Figure Q4.7 is to be modelled and analyzed. It consists of a mixing tank, mixing pipe, and CSTR. Information for modelling is given below.

    (i) Both tanks are well mixed and have constant volume and temperature.

    (ii) All pipes are short and contribute negligible transportation delay.

    (iii) All flows are constant, and all densities are constant.

    (iv) The first tank is a mixing tank.

    (v) The mixing pipe has no accumulation, and the concentration C_{A3} is constant.

    (vi) The second tank is a CSTR with A $\rightarrow$ products and $r_A = -kC_A^{3/2}$.

    (*a*) Derive a linear(ized) model (algebraic or differential equation) relating $C'_{A2}(t)$ to $C'_{A0}(t)$.

    (*b*) Derive a linear(ized) model (algebraic or differential equation) relating $C'_{A4}(t)$ to $C'_{A2}(t)$.

    (*c*) Derive a linear(ized) model (algebraic or differential equation) relating $C'_{A5}(t)$ to $C'_{A4}(t)$.

    (*d*) Combine the models in parts (i) to (iii) into one equation relating C'_{A5} to C'_{A0} using Laplace transforms. Is the response unstable? Is the response periodic?
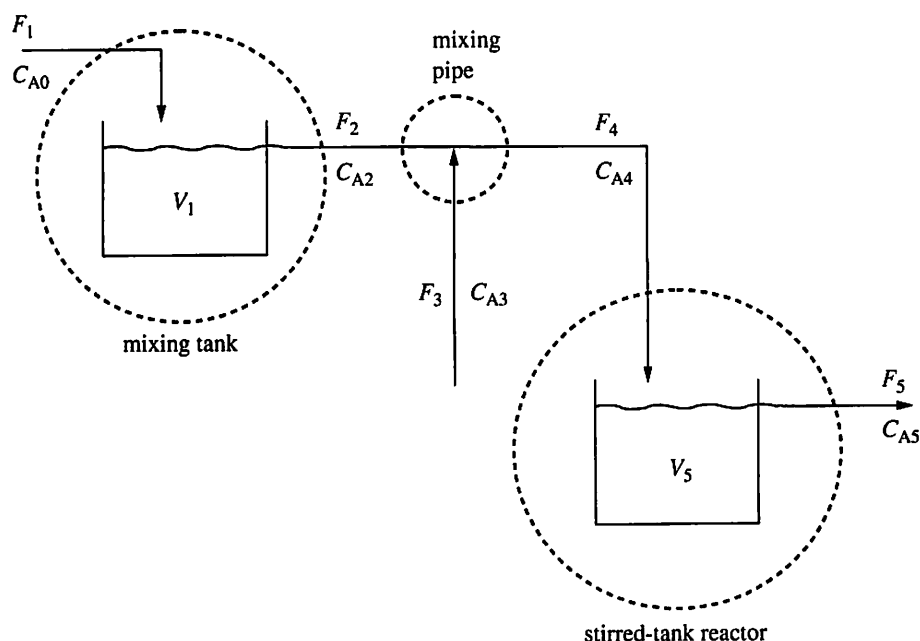

FIGURE Q4.7

Mixing and reaction processes.

(e) Determine the analytical expression for $C'_{A5}(t)$ for a step change in the inlet concentration, i.e., $C'_{A0}(t) = \Delta C_{A0} > 0$. Sketch the behavior of $C'_{A5}(t)$ in a plot vs. time.

4.8. Consider a modified version of the system in Example 4.14 with two tanks in series, each tank volume being one-half the original single-tank volume.

(a) Determine the transfer function relating the inlet and outlet concentrations.

(b) Calculate the amplitude ratio of the inlet and outlet concentration for the frequency response using equation (4.72).

(c) Determine whether either of the two designs is better (i.e., always provides the smaller amplitude ratio), for all frequencies. Explain your answer and discuss how this analysis would be used in equipment sizing.

4.9. The responses of the two levels in Figure Q4.9 are to be determined. The system is initially at steady state, and a step change is made in F_0. Assume that F_0 is independent of the levels, that the flows F_1 and F_2 are proportional to the pressure differences between the ends of the pipes, and that P' is constant. Solve for the dynamic response of both levels.
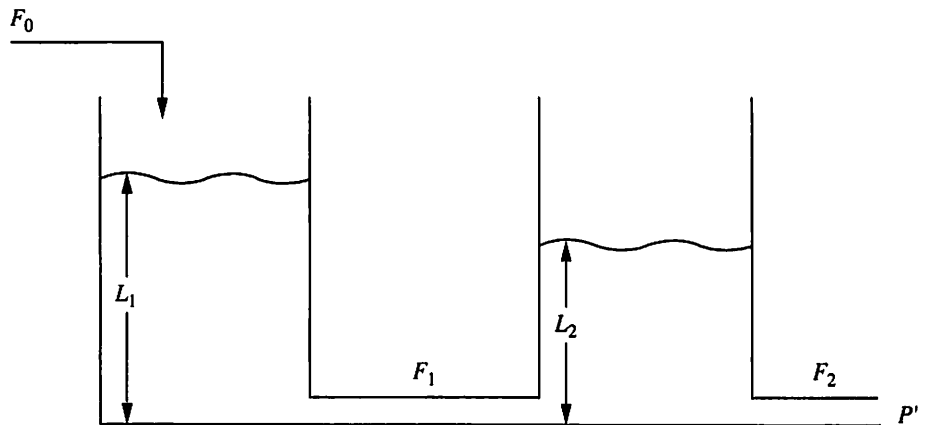

FIGURE Q4.9

4.10. For each of the block diagrams in Figure Q4.10, derive the overall input-output transfer function $X_1(s)/X_0(s)$. (Note that they are two of the most commonly occurring and important block diagrams used in feedback control.)

4.11. The isothermal chemical reactor in Figure Q4.11 includes a liquid inventory in which the turbulent flow out depends on the liquid level. The chemical reaction is first-order with negligible heat of reaction, $A \rightarrow B$, and it occurs only in the tank, not in the pipe. The system is initially at steady state and experiences a step change in the inlet flow rate, with the inlet concentration constant.

(a) Derive the overall and component material balances.

(b) Linearize the equations and take the Laplace transforms.

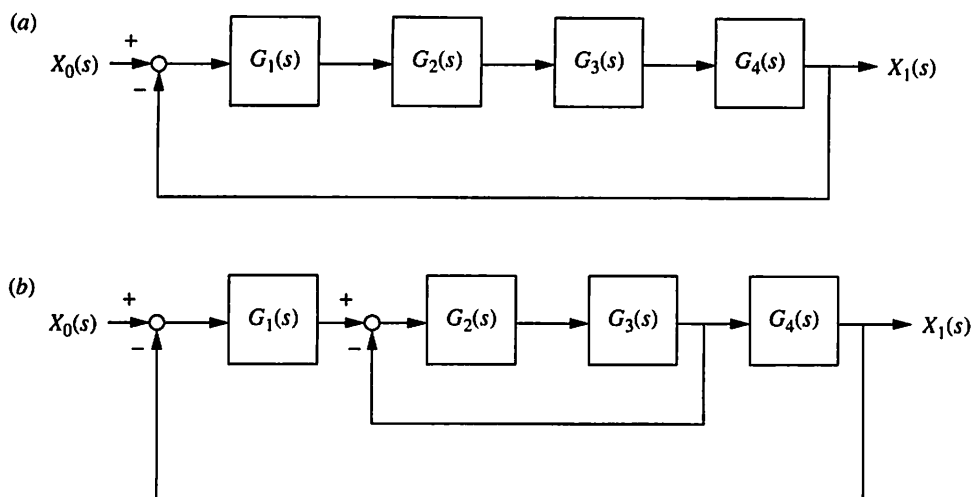(c) Determine the transfer function for $C_A(s)/F_0(s)$.

(a)

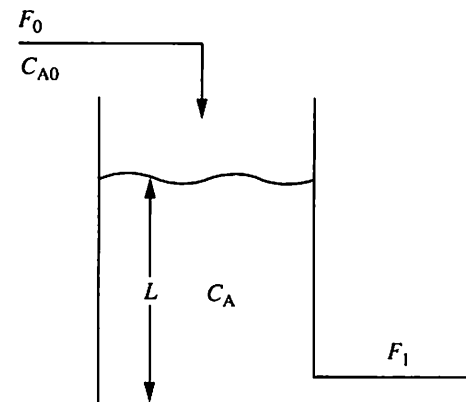**(b)**

FIGURE Q4.10

FIGURE Q4.11

4.12. The frequency response of a system can be determined empirically by introducing a sine to an input variable, waiting until the initial transient is negligible, and measuring the input and output amplitudes and the phase angle (see Figure 4.9). If this procedure were performed for several input frequencies, how could you determine whether the real physical system were first-order or second-order? After selecting the proper transfer function order, how could you determine the unknown parameters, gain, and time constant(s)? Also, discuss possible limitations to this empirical method.

4.13. A single, isothermal, well-mixed, constant-volume CSTR is considered in this question. The chemical reaction is

$$A \rightleftharpoons B$$

which is first-order with the forward and reverse rate constants k_1 and k_2, respectively. Only component A appears in the feed. The system is initially at steady state and experiences a step in the concentration of A in the feed. Formulate a model to describe this system, and solve for the concentrations of A and B in the reactor.

4.14. Answer the following questions.

(a) The initial value of a variable can be determined in a manner similar to the final value. Derive the general expression for the initial value.

(b) The transfer function in equation (4.46) can be inverted to give

$$\frac{C_{A0}(s)}{C_A(s)} = \frac{\tau s + 1}{K_p}$$

Discuss whether this is also a transfer function describing the process.

(c) The transfer function is sometimes referred to as the *impulse response* of the (linear) system. Demonstrate why this statement is true.

(d) If only the input-output relationship is required, why are all equations for the system included in the model, rather than only those equations involving the input and output variables?
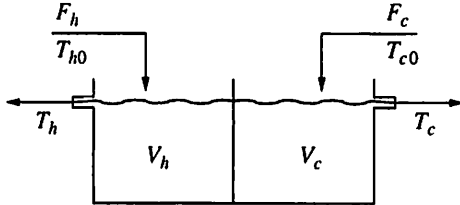
FIGURE Q4.15

4.15. A heat exchanger would be difficult to model, because of the complex fluid mechanics in the shell side. To develop a simple model, consider the two stirred tanks in Figure Q4.15, in which heat is transferred through the common wall, with $Q = UA(\Delta T)$ and UA being constant.

(a) Using typical assumptions for the stirred tanks and ignoring energy accumulation effects of the walls, derive an unsteady-state energy balance for the temperatures in both tanks.

(b) Solve for the analytical expression for both temperatures in response to a step in T_{h0}.

(c) Is it possible for this system to have periodic behavior?

4.16. For the series of isothermal CSTRs in Example 3.3:

(a) Derive the transfer function for $C_{A2}(s)/F(s)$.

(b) Use this result to determine the response of C_{A2} to an impulse in the feed rate F.

4.17. The system in Figure Q4.17 has a flow of pure A to and from a draining tank (without reaction) and a constant flow of B. Both of these flows go to an isothermal, well-mixed, constant-volume reactor with A + B → products and $r_A = r_B = -kC_AC_B$. Make any additional assumptions in determining analytical expressions for the dynamic responses from an initial steady state.

(a) Determine the flow of A to the chemical reactor in response to a flow step *into* the draining tank.

(b) Determine the concentration of A in the chemical reactor in response to (a).
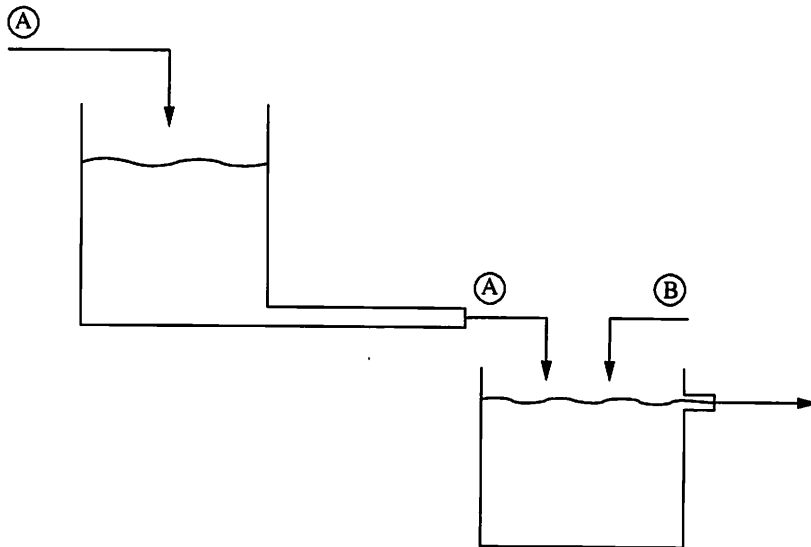

FIGURE Q4.17

4.18. The process in Figure Q4.18 involves a continuous-flow stirred tank with a mass of solid material. The assumptions for the system are:

   (1) The tank is well mixed.

   (2) The physical properties are constant, and $C_v \approx C_p$.

   (3) V = constant, F = constant [vol/time].

   (4) The solid material contributes a significant portion of the energy storage, and the temperature is uniform throughout the solid.

   (5) The heat transfer from the liquid to the metal is $UA(T - T_m)$.

   (6) Heat losses are negligible.

   (7) All variables are initially at steady state.

  (*a*) Determine the fundamental model equations that relate the behavior of $T(t)$ as $T_0(t)$ changes.

  (*b*) Derive the Laplace transform $T'(s)$ as a function of $T_0'(s)$. This involves the linear(ized) deviation variables. Identify the time constants and gains.

  (*c*) Draw a block diagram of the system of equations and derive the transfer function $T(s)/T_0(s)$.

  (*d*) State whether the system is stable or unstable and periodic or nonperiodic, and explain your answer.

  (*e*) Solve the equations and sketch the dynamic response of $T'(t)$ for a step change in $T_0'(t)$.

  (*f*) Describe briefly how the results in steps (*c*) through (*e*) would change as $UA \rightarrow \infty$.
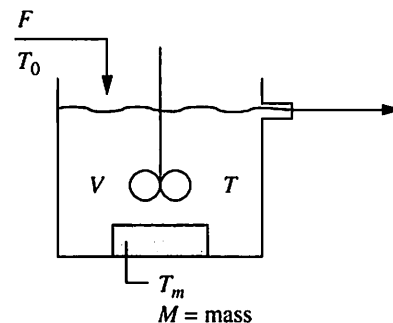

M = mass

FIGURE Q4.18