Analysis of Digital Control Systems

APPENDIX

L

L.1 INTRODUCTION

Most feedback control in the chemical process industries is currently implemented using digital computers. While most key features of control engineering are the same for continuous and digital control, some unique features of digital control should be considered. Therefore, the basic concepts of digital control were introduced in Chapter 11, and digital forms of common control algorithms are provided in Chapters 11 (PID), 12 (filtering and windup), 15 (feedforward), 21 (decoupling), and 23 (DMC). The reader is encouraged to review this material, especially the introductory material in Chapter 11, before proceeding to study this appendix.

In this appendix, we present rigorous methods, based on the z-transform, for analyzing a digital control system. As shown in Figure L.1, the z-transform enables the engineer to combine a continuous process and digital controller into one transfer function model. As with continuous systems, we can use the transfer function model to determine important properties of the system, such as its stability, final value, and frequency response. This appendix begins with an introduction to z-transforms for digital systems, which are analogous to Laplace transforms for continuous systems. Then, the application of z-transforms for control system analysis is presented. Finally, these analysis methods are applied to determine key results for PID and IMC closed-loop systems.

L.2 THE Z-TRANSFORM

The digital controller has no information on the continuous controlled variable; it has only sampled values of the controlled variable. Therefore, our analysis

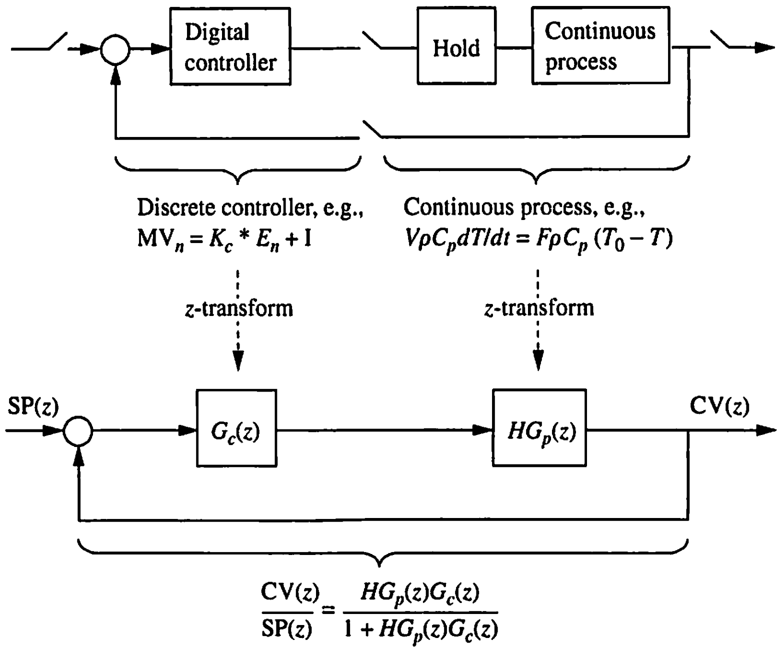


FIGURE L.1

Schematic of how z-transforms are used to combine sampled values from a continuous process and digital calculations.

approach should represent this situation. The z-transform is defined for a series of values as follows:

$$z\text{-transform:} \quad Z(Y_0, Y_1, Y_2, \ldots) = \sum_{n=0}^{\infty} Y_n z^{-n} \tag{L.1}$$

The capital "Z" denotes the z-transform, and Y_n indicates the values of the sampled variable "Y". When we consider the z-transform of a continuous variable (such as flow or temperature), we will mean the z-transform of the sampled values of the variable.

Important properties and conventions for the z-transform are summarized in the following.

1. The sampled values for a variable are assumed zero for $n < 0$.
2. The "z" variable can take complex values.
3. In this presentation, the z-transform of the sampled variable Y_n is designated by its argument, as in $Y(z)$.
4. The z-transform is a linear operator, because it satisfies the additivity and proportionality criteria

$$Z\{aY_1 + bY_2\} = aZ\{Y_1\} + bZ\{Y_2\}$$

5. A table of z-transforms and their inverses is provided in Table L.1. These pairs are unique.

6. The z-transform carries no explicit information about its sample period, although the period is known from the data collection procedure.

> We assume that the sample period (Δt) is constant for a set of sampled variables.

This assumption is valid for the vast majority of process control systems. To achieve a constant execution period, the control computer must have excess computing capacity. One method for ensuring excess capacity is to limit the number of algorithms executed per second by one processor. Also, the software must ensure that a user-written program does not exceed a maximum

TABLE L.1

Table of z-transform pairs.

No.	$G(s)$	$G(z)$
1	1 (impulse)	11
2	$\frac{1}{s}$ (step or constant)	$\frac{1}{1-z^{-1}}$
3	$\frac{1}{s+a}$	$\frac{1}{1-e^{-a(\Delta t)}z^{-1}}$
4	$\frac{1}{s^2}$	$\frac{\Delta t z^{-1}}{(1-z^{-1})^2}$
5	$\frac{1}{(s+a)(s+b)}$	$\frac{1}{b-a}\left[\frac{1}{1-e^{-a(\Delta t)}z^{-1}} - \frac{1}{1-e^{-b(\Delta t)}z^{-1}}\right]$
6	$\frac{1}{(s+a)^2}$	$\frac{(\Delta t)e^{-a(\Delta t)}z^{-1}}{(1-e^{-a(\Delta t)}z^{-1})^2}$
7	$\frac{s+a_0}{(s+a)(s+b)}$	$\frac{1}{b-a}\left[\frac{(a_0-a)}{1-e^{-a(\Delta t)}z^{-1}} - \frac{(a_0-b)}{1-e^{-b(\Delta t)}z^{-1}}\right]$
8	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\left[\frac{z^{-1}\sin(a\Delta t)}{1-2z^{-1}\sin(a\Delta t)+z^{-2}}\right]$
9	$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab(1-z^{-1})} + \frac{1}{a(a-b)(1-e^{-a(\Delta t)}z^{-1})} + \frac{1}{b(b-a)(1-e^{-b(\Delta t)}z^{-1})}$
10	$\frac{s+a_0}{s(s+a)(s+b)}$	$\frac{a_0}{ab(1-z^{-1})} + \frac{(a_0-a)}{a(a-b)(1-e^{-a(\Delta t)}z^{-1})} + \frac{(a_0-b)}{b(b-a)(1-e^{-b(\Delta t)}z^{-1})}$
11	$\frac{s+a_0}{s(s+a)^2}$	$\frac{a_0}{a_2}\left[\frac{1}{1-z^{-1}} - \frac{1}{(1-e^{-a(\Delta t)}z^{-1})} - \frac{(a/a_0)(a_0-a)(\Delta t)e^{-a(\Delta t)}z^{-1}}{(1-e^{-a(\Delta t)}z^{-1})^2}\right]$
12	$G(s)e^{-\theta s}$	$G(z)z^{-i}$ where $i = \theta/\Delta t$ = integer

Notes: Constants a, b, and a_0 are real and distinct.
Δt is the sample period.
s is the Laplace variable.
The z-transform does *not* include a zero-order hold.

allowable computing time; limited computing times are enforced by a monitoring program that interrupts a program exceeding the maximum time.

With a constant sample period, the sampled data can be represented as

$$Y_n = Y(n\Delta t)$$

where n is the sample number and Δt is the sample period.

7. To reiterate, no information on intersample values is available from the z-transform.

EXAMPLE L.1.

Sampled values of a temperature are provided. Determine the first few terms in the z-transform of this sampled variable. The sample period, $\Delta t = 10$ seconds.

$$\begin{aligned} T(0) &= 310 \\ T(\Delta t) &= 312 \\ T(2\Delta t) &= 315 \\ T(3\Delta t) &= 318 \end{aligned}$$

The sampled values can be substituted directly into equation (L.1) to give the following:

$$Z(T(n\Delta t)) = 310 + 312z^{-1} + 315z^{-2} + 318z^{-3} + \cdots$$

EXAMPLE L.2.

Use the data given in Example L.1 to develop the z-transform of the temperature with a sample period of 5 seconds.

The data has no information about intersample behavior. Therefore, we cannot determine the sampled values at 5, 15, ..., seconds. We cannot determine the z-transform for 5-second samples from the data provided.

Next, we will evaluate the z-transforms for the sampled values of several common variables.

UNIT STEP INPUT. $U(n\Delta t) = 1 \quad \text{for all } n \geq 0$

$$Z(U(n\Delta t)) = Z(1, 1, 1, 1, \ldots) = \sum_{n=0}^{\infty}(1)z^{-n} = \sum_{n=0}^{\infty} z^{-n} \tag{L.2}$$

Using the relationship that $\sum_{n=0}^{\infty} z^{-n} = 1/(1 - z^{-1})$ for $|z| > 1$, we obtain for following result:

$$Z(U(n\Delta t)) = \frac{1}{1 - z^{-1}} \tag{L.3}$$

UNIT IMPULSE. $Y(0) = 1$ and $Y(n\Delta t) = 0 \quad \text{for } n > 0$

$$Z(1, 0, 0, 0, \ldots) = (1)z^{-0} + 0z^{-1} + 0z^{-2} \cdots + = 1 \tag{L.4}$$

RAMP. $Y(t) = at$ so that $Y(n\Delta t) = an\Delta t$

$$Z(0, a\Delta t, 2a\Delta t, \ldots) = \sum_{n=0}^{\infty}(an\Delta t)z^{-n}$$
$$= 0 + (a\Delta t)z^{-1} + (2a\Delta t)z^{-2} + (3a\Delta t)z^{-3} + \cdots$$
$$= (a\Delta t)z^{-1}(1 + 2z^{-1} + 3z^{-1} + \cdots)$$
$$Z(0, a\Delta t, 2a\Delta t, \cdots) = \frac{(a\Delta t)z^{-1}}{(1 - z^{-1})^2} \tag{L.5}$$

The last step relied on the following relationship:

$$1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + \cdots + = \frac{1}{(1 - z^{-1})^2} \quad \text{for } |z| > 1$$

TRANSLATION (DEAD TIME). $y(n\Delta t - i\Delta t)$ where i = integer number of samples in the dead time

$$Z(Y(n\Delta t - i\Delta t) = \sum_{n=0}^{\infty} Y(n\Delta t - i\Delta t)z^{-n} \quad \text{substituting } k = n - i$$
$$= \sum_{k=-i}^{\infty} Y(k\Delta t)z^{-(k+i)} \quad \text{with } Y(k\Delta t) = 0 \text{ for } k > 0$$
$$= \sum_{k=0}^{\infty} Y(k\Delta t)z^{-i}z^{-k}$$
$$= z^{-i}\sum_{k=0}^{\infty} Y(k\Delta t)z^{-k}$$
$$Z(y(n\Delta t - i\Delta t) = z^{-i}Y(z) \tag{L.6}$$

Therefore, the z-transform of a variable with dead time is the product of z to the power of $-i$, (where i is the number of samples in the dead time) and the z-transform of the function without dead time. Note that this development relied on the dead time being an integer value.

> In this appendix, we will assume that the dead time is an integer multiple of the sample time, i.e., $i = \theta/\Delta t$ = integer.

For extensions when the dead time is not an integer, see material on modified z-transforms in Smith (1972).

DIFFERENCE EQUATIONS. Calculations performed for control (including controller, filter, and models) take the form of difference equations in digital computers; therefore, we need to take the z-transform of such equations. The control equation uses current and past sampled values of variables; naturally, future values are not available. The following expressions give the z-transform for current $Y(n)$ and past (Y_{n-i}) values of a sampled variable:

$$Z(Y_n) = Z(Y(n\Delta t), Y((n-1)\Delta t), Y((n-2)\Delta t), \ldots) = Y(z) \qquad \text{(L.7a)}$$

$$Z(Y_{n-i}) = Z(Y((n-i)\Delta t), Y((n-i-1)\Delta t), Y((n-i-2)\Delta t), \ldots) = z^{-i}Y(z) \qquad \text{(L.7b)}$$

We will be deriving controller calculations in the form of z-transforms and would like to implement these controllers in a digital computer as difference equations. Therefore, we will be applying the inverse of equations (L.7*a*) and (L.7*b*), for example,

$$Z^{-1}(Y(z)) = Y_n \qquad Z^{-1}(z^{-i}Y(z)) = Y_{n-i}$$

From the above expressions we see why z^{-1} is similar to the *backward shift operator*, because z^{-i} indicates that the variable in a difference equation is i samples "back" from the current variable.

INTEGRAL. The integral mode in the PID controller is calculated in the digital computer using a numerical approximation based on sampled values. As described in Chapter 11, the discrete form of the integral mode using rectangular integration is given by the following expression, with E representing the error between the set point and measured controlled variable,

$$\int_0^t E(t)\,dt \approx \sum_{i=0}^{n} (\Delta t)E_i$$

We can take the z-transform of the expression, applying the expressions for the difference equations to give

$$Z\left(\sum_{i=0}^{n}(\Delta t)E_i\right) = (\Delta t)E(z)\sum_{i=0}^{n} z^{-i}$$

For large values of n, $\sum_{i=0}^{n} z^{-i} \approx 1/(1-z^{-1})$ giving the expression for the z-transform of rectangular integration.

$$Z\left(\sum_{i=0}^{n}(\Delta t)E_i\right) = \frac{(\Delta t)E(z)}{1-z^{-1}} \qquad \text{(L.8)}$$

DERIVATIVE. The derivative mode in the PID controller is calculated in the digital computer using a numerical approximation based on sampled values. As described in Chapter 11, a common discrete approximation of the derivative mode is given by the backward difference, with CV representing the measured controlled variable

$$\left(\frac{d\,\text{CV}}{dt}\right)_{t=n\Delta t} \approx \frac{\text{CV}_n - \text{CV}_{n-1}}{\Delta t}$$

The z-transform can be evaluated to yield the z-transform of the derivative.

$$Z\left(\frac{\text{CV}_n - \text{CV}_{n-1}}{\Delta t}\right) = \frac{1}{\Delta t}(\text{CV}(z) - z^{-1}\text{CV}(z)) = \left(\frac{1-z^{-1}}{\Delta t}\right)\text{CV}(z) \qquad \text{(L.9)}$$

FIRST-ORDER DIGITAL FILTER. A first-order filter can be used to reduce noise in a measurement prior to the control calculation. The digital filter discussed in Chapter 12 is a discrete form of the continuous filter and is repeated below for X as the input and Y as the output, with the filter time constant, τ_F.

$$Y_n = \alpha Y_{n-1} + (1-\alpha)X_n \qquad \text{with } \alpha = e^{-\Delta t/\tau}$$

The z-transform of this difference equation gives

$$Y(z) = \alpha z^{-1} Y(z) + (1-\alpha)X(z)$$
$$= \frac{(1-\alpha)}{(1-\alpha z^{-1})} X(z) \tag{L.10}$$

FINAL VALUE THEOREM. The final value of the error is important, because if it is zero, the control system returns the controlled variable to its set point after a disturbance or set point change. Thus, we introduce the final value theorem that provides a direct manner for evaluating the final value of a sampled variable from its z-transform. We begin by stating the theorem and proceed to prove the expression.

$$\lim_{n\to\infty} Y(n\Delta t) = \lim_{z\to 1} (1-z^{-1})Y(z) \tag{L.11}$$

$$= \lim_{z\to 1}(1-z^{-1}) \sum_{n=0}^{\infty} Y(n\Delta t) z^{-n}$$

$$= \lim_{z\to 1} \sum_{n=0}^{\infty} (Y(n\Delta t) - z^{-1} Y(n\Delta t)) z^{-n}$$

$$= \lim_{z\to 1} \sum_{n=0}^{\infty} Y(0) + (Y(\Delta t) - Y(0))z^{-1} + (Y(2\Delta t) - Y(\Delta t))z^{-2} + \cdots$$

$$= \lim_{n\to\infty} Y(n\Delta t)$$

The last step results from the cancellation of all but the last term in the series. Note that this expression is valid only when the system in stable, so that the terms $Y(n\Delta t)$ approach the same value as the sample number n becomes large.

INVERSE z-TRANSFORM. We would like to evaluate the inverse of the z-transform to determine the *sampled*, time-domain values of the variable. We will present two methods in this appendix: (1) long division to reinforce the principles and (2) partial fractions to provide the basis for important generalizations.

Before covering these methods, we note the following important feature of z-transforms:

The z-transform is always a ratio of two polynomials in z.

The structure of the Laplace transform was more complex because of the dead time ($e^{-\theta s}$); however, the dead time in digital systems (z^{-i}) simply increases the order of polynomials. Also, because real processes involve differential equations, the order of the denominator is greater than that of the numerator (after the dead time is factored out). The methods of inversion take advantage of the polynomial structure of the z-transforms.

One method of inverting a z-transform is long division, which provides a series expression in powers of z^{-1}. The sampled values can be evaluated by comparing the terms in the series with the definition of the z-transform.

EXAMPLE L.3.

Evaluate the inverse z-transform for the following expression, and evaluate the final value.

$$Y(z) = \frac{0.6z^{-1}}{1 - 1.8z^{-1} + 0.8z^{-2}}$$

The expression can be expanded by long division to give

$$\begin{array}{r|l} & 0.6z^{-1} + 1.08z^{-2} + 1.46z^{-3} + 1.77z^{-4} \\ \hline 1 - 1.8z^{-1} + 0.80z^{-2} & 0.6z^{-1} \end{array}$$

The first few sampled values can be determined by comparing this result with the definition of the z-transform as shown in the following:

$$Y(z) = \sum_{n=0}^{\infty} Y(n\Delta t)z^{-n} = 0z^{-0} + 0.6z^{-1} + 1.08z^{-2} + 1.46z^{-3} + 1.77z^{-4} + \cdots$$

Therefore, $Y(0) = 0$, $Y(\Delta t) = 0.60$, $Y(2\Delta t) = 1.08$, $Y(3\Delta t) = 1.46$, and $Y(4\Delta t) = 1.77$. The final value can be determined by applying the final value theorem.

$$\lim_{z\to 1}(1 - z^{-1})\left(\frac{0.60z^{-1}}{1 - 1.8z^{-1} + 0.80z^{-2}}\right) = \lim_{z\to 1}(1 - z^{-1})\left(\frac{0.60z^{-1}}{(1 - z^{-1})(1 - 0.8z^{-1})}\right) = 3.0$$

The second method for inverting z-transforms uses partial fractions to represent a complex expression by the sum of several simpler expressions. Each of the simpler expressions can be inverted using Table L.1 or by long division. Thus, we can invert essentially any z-transform of a realistic process variable using this approach. In addition, we can easily determine the stability of a variable.

The partial fraction method is summarized in the following:

$$Y(z) = \frac{N(z)}{D(z)} = \frac{C_1}{(1 - p_1 z^{-1})} + \frac{C_2}{(1 - p_2 z^{-1})} + \cdots \qquad \text{(L.12)}$$

where
$Y(z)$ = z-transform of the output variable
$N(z)$ = numerator polynomial in z of order m
$D(z)$ = denominator polynomial in z or order n
p_i = roots of the equation $D(z) = 0$, also called the *poles* (distinct roots assumed here)

The partial fractions method requires that the order of the denominator be greater than the order of the numerator, i.e., $n > m$, after the dead time is factored out. This requirement will be satisfied by models encountered in process control, after the dead time is temporarily removed. Initially, the C_i's are unknowns in equation (L.11) and must be determined so that the equation is satisfied. The partial fraction expansions and the resulting Heaviside expansion formula are presented in Appendix H for evaluating the constants, so the details are not repeated here. Suffice to say that the same procedures can be applied to evaluate the constants C_i here as well.

One important stability result can now be presented, because equation (L.12) shows that all z-transforms can be represented as a sum of simpler expressions. Let us expand one of the terms in Equation (L.12) by long division.

$$Y(n\Delta t) = C_i Z^{-1}\left(\frac{1}{1 - p_i z^{-1}}\right) = C_i Z^{-1}(1 + p_i z^{-1} + p_i^{-2} z^{-1} + p_i^{-3} z^{-1} + \cdots) \tag{L.13}$$

By comparing the equation above with the definition of the z-transform, the sampled valued can be determined to be

$$Y(0) = C_i, Y(\Delta t) = C_i p_i, Y(2\Delta t) = C_i p_i^2, Y(3\Delta t) = C_i p_i^3, \ldots$$

Clearly, the sampled variables will be stable if $p_i \leq 1.0$ and will be unstable (increase toward $\pm\infty$) if $p_i > 1.0$. We generalize this result to a test for stability of a sampled data variable in the following:

- Determining the stability for a z-transform with *distinct roots*

Stable	$\lvert p_i \rvert \leq 1.0$
Unstable	$\lvert p_i \rvert > 1.0$

- For *repeated and complex roots*, the result is similar:

Stable	$\lvert p_i \rvert < 1.0$
Unstable	$\lvert p_i \rvert \geq 1.0$

The roots of higher-order polynomials are difficult to evaluate by hand calculation, but numerical methods are available and standard algorithms can be used in software such as MATLAB™.

EXAMPLE L.4.

Determine whether the following variable is stable:

$$Y(z) = \frac{0.6z^{-1}}{1 - 1.8z^{-1} + 0.8z^{-2}} = \frac{0.6z^{-1}}{(1 - z^{-1})(1 - 0.8z^{-1})}$$

The roots of the denominator are 1.0 and 0.80. Since they are distinct and less than or equal to 1.0, the variable is stable. Note that this is the variable considered in Example L.3, where the final value was determined. The application of the final value theorem is valid only for stable variables.

In continuous systems, the roots of the polynomial in the denominator of the Laplace transform provided information about the damping of the variable in the time domain. This is true for the roots of the denominator of the z-transform as well. Let us evaluate the first few sampled values for some example z-transforms, and then we will generalize the results.

		First six sampled values						
z-transform	**Root**	**0**	**1**	**2**	**3**	**4**	**5**	**Comments**
$1/(1-1.1z^{-1})$	1.1	1.0	1.1	1.21	1.33	1.46	1.61	Overdamped, *unstable!*
$1/(1-0.9z^{-1})$	0.90	1.0	0.90	0.82	0.73	0.66	0.59	Overdamped, slow response
$1/(1-0.5z^{-1})$	0.50	1.0	0.50	0.25	0.125	0.063	0.032	Overdamped, faster response
$1/(1+0.9z^{-1})$	−0.90	1.0	−0.90	0.81	−0.73	0.66	−0.59	Highly oscillatory, *ringing*
$1/(1+1.1z^{-1})$	−1.1	1.0	−1.1	1.21	−1.33	1.46	−1.61	Oscillatory, *unstable!*

Note that the real pole greater than +1.0 is unstable. Also, the positive real poles with magnitudes less than 1.0 give stable, overdamped responses. Finally, poles with real parts near −1.0 result in highly oscillatory responses. If the magnitude is less than 1.0, the oscillations damp out; this is behavior is termed *ringing*. If the magnitude of the pole is greater than 1.0, the response is unstable.

The results on stability and damping are often summarized in graphical displays of the roots of the denominator in which the real and complex parts of the roots are plotted as shown in Figure L.2. Therefore, the *unit circle* is plotted for easy reference, since roots inside the unit circle yield stable performance. The following guidelines are often used:

- Stable systems have all roots of the denominator (the poles) within the unit circle. Any root outside of the unit circle results in instability.
- Roots near the origin represent faster dynamics than roots far from the origin, i.e., near the unit circle.
- Roots with real parts near −1.0 result in highly oscillatory, ringing behavior.

Rules regarding "good" pole locations have been suggested (e.g., Franklin et al., 1990), but these rules are limited to second-order systems with a constant numerator term. As we complete the presentation in this appendix, we will see that realistic systems have terms (powers of z) in the numerator as well as the denominator, and systems can be of much higher than second-order. Therefore, analysis of the dynamic behavior of digital control (beyond the general guidelines above) should be performed using dynamic simulation, which is straightforward for linear systems.

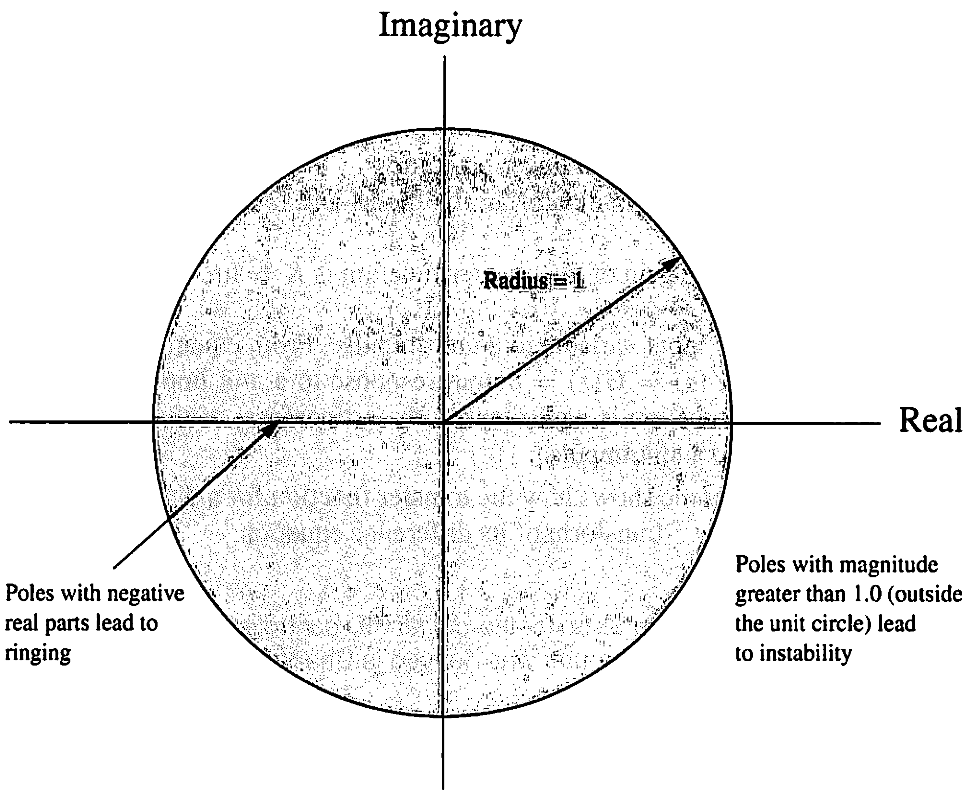


FIGURE L.2

The roots of the z-transform denominator plotted in the complex plane. If all roots are inside the unit circle, the sampled values are stable.

L.3 ▫ METHODS FOR ANALYZING DIGITAL CONTROL SYSTEMS

In this section, we introduce methods for analyzing linear, closed-loop digital control systems. As with continuous systems, the analysis is based on transfer function models and block diagrams, and the results are the three key features of a linear system that can be determined without complete solution of the transient response, stability, final value, and frequency response. We begin by defining the transfer function for input X and output Y.

$$\text{Transfer function:} \quad G(z) = \frac{Y(z)}{X(z)} \tag{L.14}$$

The following assumptions are associated with the transfer function:

1. The initial conditions for X and Y are zero. Building models in deviation variables from an initial steady state easily satisfies this condition.
2. The samples of both variables are at the same period, are synchronized, and are instantaneous. These assumptions are valid for essentially all chemical

processes, because electronic analog-digital conversion and signal sampling devices are very fast in comparison to the dynamics of the process equipment.

Some properties of the transfer function are stated here.

(*a*) The transfer function is a linear operator.
(*b*) The roots of the denominator are the poles and indicate the stability of the variable.
(*c*) The steady-state gain of the transfer function is $K = \lim_{z \to 1} G(z)$, when $G(z)$ is stable.
(*d*) Let us set the input variable to a unit impulse. From equation (L.4), $X(z) = 1.0$, so that $Y(z) = G(z) =$ output response to a unit impulse input. Thus, the transfer function is equivalent to the z-transform of the sampled output responding to a unit impulse.

The next example shows how the transfer function for a digital calculation is found by taking the z-transform of its difference equation.

EXAMPLE L.5.

Determine the transfer function for the *digital* PID controller. The discrete equation executed in the digital computer was derived in Chapter 11 and is repeated here.

$$MV'_n = K_c \left(E'_n + \frac{\Delta t}{T_I} \sum_{i=0}^{n} E'_i \right)$$

Note that the equation is in deviation variables. The z-transform of the equation can be taken using the relationships in equations (L.7) and (L.8).

$$MV'(z) = K_c \left(1 + \frac{\Delta t}{T_I} \frac{1}{1 - z^{-1}} \right) E'(z)$$

This can be rearranged to form the transfer function, with the prime dropped by convention, because the transfer function is always in terms of deviation variables.

$$\text{PI controller:} \quad G_c(z) = \frac{MV(z)}{E(z)} = K_c \left(1 + \frac{\Delta t}{T_I} \frac{1}{1 - z^{-1}} \right) \tag{L.15}$$

The transfer functions for the following controllers can be derived by similar development:

$$\text{P-only controller:} \quad G_c(z) = \frac{MV(z)}{E(z)} = K_c \tag{L.16}$$

$$\text{PID controller:} \quad G_c(z) = \frac{MV(z)}{E(z)} = K_c \left[1 + \frac{\Delta t}{T_I} \frac{1}{1 - z^{-1}} + \frac{T_d}{\Delta t}(1 - z^{-1}) \right] \tag{L.17}$$

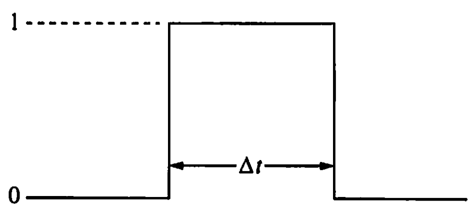


FIGURE L.3

Dynamic response of a zero-order hold to an input of magnitude 1.0.

Now we turn to modelling the sampled values from a continuous process. As explained in Chapter 11, the signal from a digital controller is converted to a continuous signal in a digital-to-analog (D/A) converter, and the analog signal is held constant between samples by a zero-order hold. The time behavior of a zero-order hold is shown in Figure L.3, which shows that the value is unchanged over the first Δt after the digital controller has calculated the output value, and then the zero-order hold decreases to zero for the past controller output. Recall that the digital controller produces an updated controller output, so that the signal to the valve changes immediately to the new value of the controller output. The dynamic behavior of a sampled system is given in Figure L.4 to clearly show the sampled and continuous variables.

The zero-order hold has the time behavior of a pulse function. The Laplace transform of a pulse function is derived in equation (4.9) and is repeated in the following, with $C = \Delta t$ so that the integral is 1.0, as shown in Figure L.3.

$$\text{Zero-order hold:} \qquad H(s) = \frac{1 - e^{-(\Delta t)s}}{s}$$

The symbol $H(s)$ is used for consistency with other publications.

Next, we proceed to determine the transfer function model of the system in Figure L.4, which represents the continuous components of a digital control system, along with the interfacing components (A/D, D/A, and hold). The Laplace transform of the zero-order hold with a series process is given in the following:

$$H(s)G(s) = \left(\frac{1 - e^{-(\Delta t)s}}{s}\right) G(s) = (1 - e^{-(\Delta t)s})\left(\frac{G(s)}{s}\right)$$

The term $e^{-(\Delta t)s}$ is the Laplace transform of a dead time of duration Δt. Thus, the z-transform of $e^{-(\Delta t)s}$ is the unit dead time z^{-1}, that is, $Z(e^{-(\Delta t)s}) = z^{-1}$. Taking the z-transform of the equation above gives

$$Z(H(s)G(s)) \equiv HG(z) = Z\left((1 - e^{-(\Delta t)s})\frac{G(s)}{s}\right) = (1 - z^{-1})Z\left(\frac{G(s)}{s}\right) \qquad \text{(L.18)}$$

Thus, the z-transform of the series hold and process can be evaluated and using equation (L.18). The term $Z(G(s)/s)$ can be determined using Table L.1. It is important to note that in general, $HG(z) \neq H(z)G(z)$! Now, let us consider a few examples.

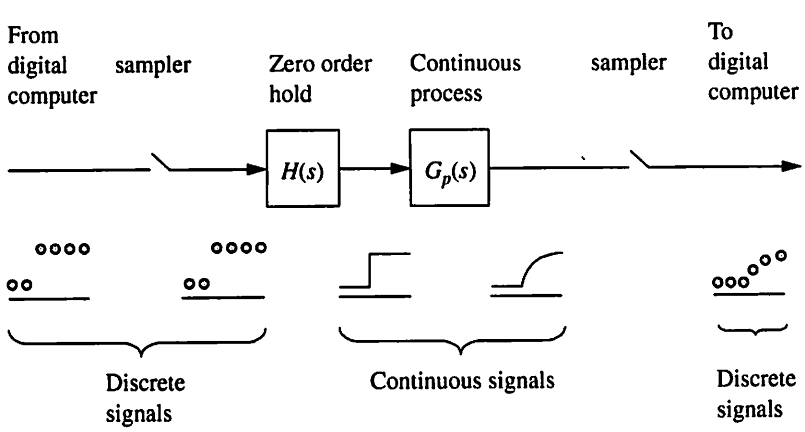


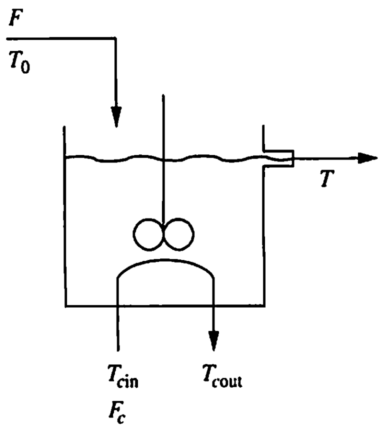
FIGURE L.4

Schematic of continuous and discrete (digital) signals for digital control of a continuous process.

EXAMPLE L.6.

Determine the transfer function for the sampled heat exchanger in Example 3.7 with a sample period of 15 seconds (0.25 minute).

The model for the heat exchanger temperature as a function of the coolant flow rate is derived in Example 3.7 and repeated here.



$$\frac{T(s)}{F_c(s)} = G(s) = \frac{-33.9}{11.9s+1} \quad \text{with time in minutes}$$

The transfer function for the sampled system with zero-hold is given in the following:

$$HG(z) = (1-z^{-1})Z\left(\frac{G(s)}{s}\right) = (1-z^{-1})Z\left(\frac{-33.9}{s(11.9s+1)}\right) = (1-z^{-1})Z\left(\frac{-33.9/11.9}{s(s+1/11.9)}\right)$$

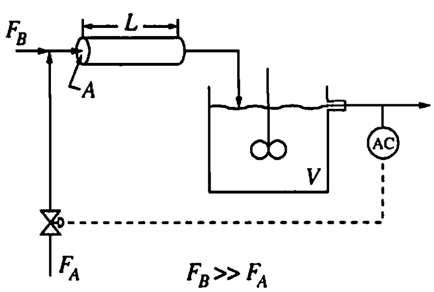
The z-transform is evaluated using entry 5 in Table L.1 (with $K = 33.9, a = 0, b = 1/\tau$) to give

$$HG(z) = (1-z^{-1})K\frac{b}{b}\frac{(1-e^{-\Delta t/\tau})z^{-1}}{(1-z^{-1})(1-e^{-\Delta t/\tau}z^{-1})} = \frac{-0.705z^{-1}}{(1-0.979z^{-1})} \qquad (L.19)$$

We can determine that the system is stable because the denominator of the transfer function has a root at $0.979 < 1.0$.

EXAMPLE L.7.

Determine the z-transform for the sampled mixing process with dead time in Chapter 9 with a sample period of 1 minute. Here, we will consider the process and hold *without control.*



The process model is repeated here

$$\frac{A(s)}{v(s)} = G(s) = \frac{1.0e^{-5s}}{(5s+1)} \quad \text{with time in minutes}$$

The transfer function for the sampled system with zero-order hold is given in the following:

$$HG(z) = (1-z^{-1})Z\left(\frac{G(s)}{s}\right) = (1-z^{-1})Z\left(\frac{1.0e^{-5s}}{s(5s+1)}\right) = (1-z^{-1})z^{-5}Z\left(\frac{1/5}{s(s+1/5)}\right)$$

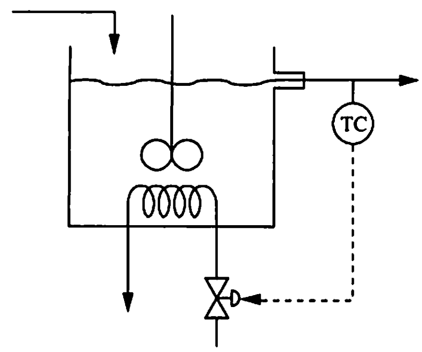
The dead time is 5 sample periods; $i = \theta/\Delta t = 5/1 = 5$. The z-transform is evaluated using entry 5 in Table L.1 (with $K = 1, a = 0$, and $b = 1/\tau$) to give

$$HG(z) = (1-z^{-1})K\frac{b}{b}z^{-5}\frac{(1-e^{-\Delta t/\tau})z^{-1}}{(1-z^{-1})(1-e^{-\Delta t/\tau}z^{-1})} = \frac{0.181z^{-6}}{1-0.819z^{-1}} \qquad (L.20)$$

Now that we can determine transfer function models for the digital calculations and the continuous process with hold, we can combine these transfer functions to describe the closed-loop behavior. As with continuous systems, we will use block diagrams to derive the overall model, and we will apply the same rules and procedures in block diagram manipulation. We consider the block diagram in Figure L.1, in which the final element and sensor have been combined with the process in $G_p(s)$. The closed-loop transfer function for the system is given in the following:

$$\frac{CV(z)}{SP(z)} = \frac{HG_p(z)G_c(z)}{1 + HG_p(z)G_c(z)} \quad \text{(L.21)}$$

EXAMPLE L.8.



Determine the transfer function for digital PI control of the stirred-tank heat exchanger in Example L.6 and evaluate key aspects of the performance. (Note that this is the same closed-loop system considered in Example 8.7 for continuous control.)

The closed-loop transfer function is determined by substituting for $HG_p(z)$ and $G_c(z)$ in equation (L.21). The general forms of the transfer functions are given in the following:

First-order process **PI controller**

$$HG_p(z) = \frac{K_p(1 - e^{-\Delta t/\tau})z^{-1}}{1 - e^{-\Delta t/\tau}z^{-1}} \qquad G_c(z) = K_c\left(1 + \frac{\Delta t}{T_I}\frac{1}{1 - z^{-1}}\right)$$

Substituting the individual transfer function models into equation (L.21) gives the following transfer function, with $B = e^{-\Delta t/\tau}$ used to simplify the notation:

$$\frac{CV(z)}{SP(z)} = \frac{\dfrac{K_pK_c(1 - B)z^{-1}\left(1 + \dfrac{\Delta t}{T_1}\dfrac{1}{1 - z^{-1}}\right)}{1 - Bz^{-1}}}{1 + \dfrac{K_pK_c(1 - B)z^{-1}\left(1 + \dfrac{\Delta t}{T_I}\dfrac{1}{1 - z^{-1}}\right)}{1 - Bz^{-1}}}$$

The stability of the system can be determined by evaluating the roots of the characteristic equation, which can be set equal to zero and rearranged to give the following:

$$0 = 1 + (-(1 + B) + K_pK_c(1 - B)(1 + \Delta t/T_I))z^{-1} + [B - K_pK_c(1 - B)]z^{-2}$$

This equation can be multiplied by z^2 to give the following quadratic equation to be solved for the roots:

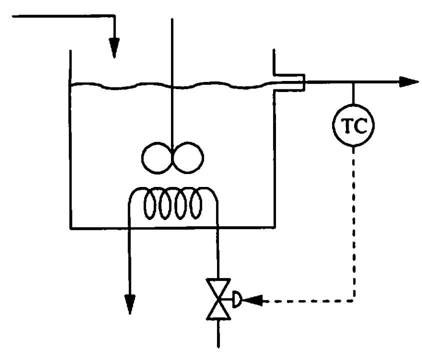
$$0 = z^2 + (-(1 + B) + K_pK_c(1 - B)(1 + \Delta t/T_I))z + [B - k_pK_c(1 - B)] \quad \text{(L.22)}$$

The parameters for this problem are (with the tuning from Example 8.7)

$$K_p = -33.9 \text{ K/(m}^3\text{/min)} \qquad K_c = -0.059 \text{ (m}^3\text{/min)/K}$$
$$\tau = 11.9 \text{ min} \qquad T_I = 0.95 \text{ min} \qquad \Delta t = 0.25 \text{ min}$$

For the parameters in this example, the roots of the characteristic equation are $0.98 \pm 0.06j$, which have a magnitude less than 1.0; therefore, the system is stable.

EXAMPLE L.9.



Evaluate the stability of the digital heat exchanger control system with PI controller and tuning from Example L.8 with different execution periods.

The stability can be determined by evaluating the roots of the characteristic equation; all roots within the unit circle, i.e., having magnitudes less then 1.0, indicate stability for step-type inputs. The characteristic equation was determined in Example L.8 and is repeated below.

$$0 = z^2 + (-(1+B) + K_pK_c(1-B)(1+\Delta t/T_I))z + [B - K_pK_c(1-B)]$$

The parameters Δt and $B = e^{-\Delta t/\tau}$ appear in the equation, so that changing the execution period (Δt) changes the roots of the equation. Numerical results are summarized below for the parameters in this example.

Execution period, Δt (min)	**Roots of the characteristic equation (min^{-1})**	**Maximum magnitude of root**	**Stability**
0.25	$0.98 \pm 0.06j$	0.98	Stable
1.0	$0.79 \pm 0.36j$	0.87	Stable
2.0	$0.44 \pm 0.58j$	0.73	Stable
3.0	$-0.38 \pm 0.57j$	0.68	Stable
3.925	$0.156, -1.009$	$1.009 > 1.0$	**Unstable**

The pole location begins in the stable and well-damped region, and as the execution period increases (with tuning unchanged), the poles move toward the ringing region (negative real parts) and ultimately to instability (outside of the unit circle). This rigorous analysis is consistent with the simulation studies and guidelines presented in Chapter 11.

Finally, we would like to evaluate the frequency response of a linear, digital system. Recall that the frequency response is the behavior of the output for a sine input after sufficient time for initial transients to damp out. For a linear system, the output behavior will be a sine with the same period as the input. The frequency response of a digital system can be determined by using the relationship that both z^{-1} and $e^{-(\Delta t)s}$ represent a unit dead time. Therefore, z^{-1} can be replaced with $e^{-(\Delta t)s}$ and the Laplace variable (s) can be set to $j\omega$, with ω being the sine frequency (Franklin et al., 1990). This approach is now applied to a digital system.

EXAMPLE L.10.

Determine the frequency responses for two first-order filters; (*a*) a continuous and (*b*) a digital. For this example, let the filter time constant (τ_f) be 0.50 second and the sample period (Δt) be 0.25 second.

The first-order filter is a first-order lag without dead time and with a steady-state gain of 1.0.

(*a*) The transfer function for the continuous filter is $G_f(s) = 1/(\tau_f s + 1)$. The frequency response can be evaluated using methods presented in Chapters 4 and 10.

Continuous filter: $$G_f(s) = \frac{1}{\tau s + 1} \qquad G_f(\omega j) = \frac{1}{\sqrt{1 + \tau_f^2 \omega^2}}$$

(*b*) The transfer function for the digital filter is given in equation (L.10), with $\alpha = e^{-\Delta t/\tau_F}$. The frequency response can be evaluated by replacing z^{-1} with $e^{-(\Delta t)\omega j}$ to give the following:

Digital filter: $$G_f(z) = \frac{1-\alpha}{1-\alpha z^{-1}} \qquad G_f(e^{(\Delta t)\omega j}) = \frac{1-\alpha}{1-\alpha e^{-(\Delta t)\omega j}}$$

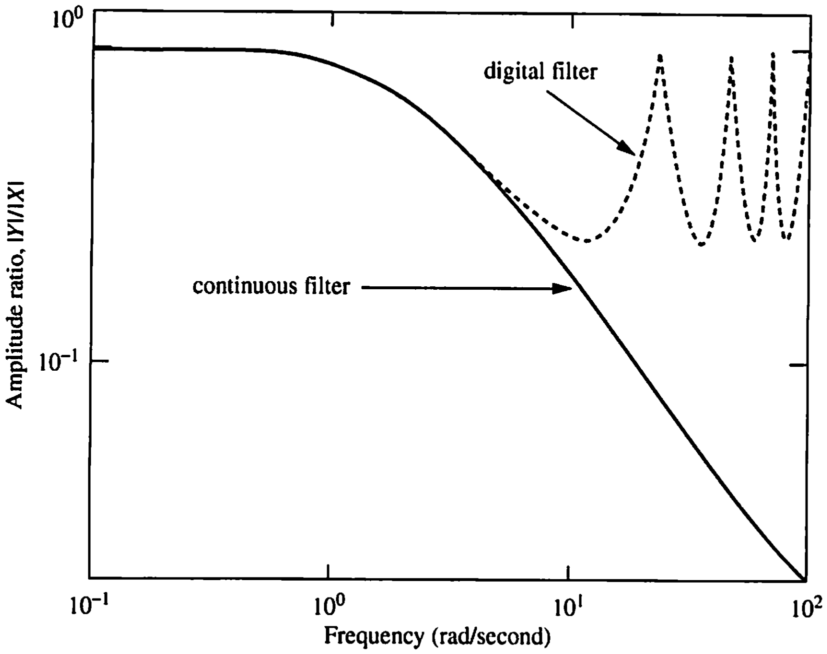


FIGURE L.5

Frequency responses for continuous and digital ($\Delta t = 0.25$ second) first-order filters. The filter time constant is 0.50 second.

The amplitude ratios, $|Y|/|X|$, for both filters are plotted in the Bode diagram in Figure L.5. One important use of the first-order filter is to attenuate higher-frequency noise from a measurement. Note that the continuous filter is effective at very high frequencies. The digital filter performs similarly to the continuous filter at low frequencies. At higher frequencies, the digital filter is not effective. Since many measurement signals contain very high frequency noise from electrical interference, each measurement signal to digital control equipment has a continuous (analog) filter with a small time constant before the measurement is converted to a digital signal.

The frequency beyond which the digital filter deviates from the continuous filter can be *estimated* using Shannon's sampling theorem; "A continuous function with all frequency components at or below ω' can be represented uniquely by values sampled at a frequency equal to or greater than $2\omega'$." Applying this approach, the highest frequency at which the digital signal closely estimates the continuous signal is (using $\omega_{\text{sample}} = \pi/\Delta t$).

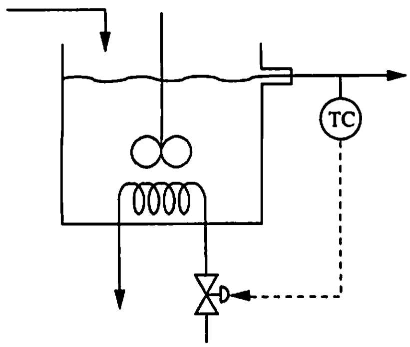
$$\omega' = \omega_{\text{sample}}/2 = \pi/(2\Delta t) = 3.14/2(0.25) = 6.28 \text{ radians/second}$$

This frequency is a reasonable estimate of the maximum frequency at which the digital filter provides a reasonable estimate of the desired continuous signal.

L.4 ▫ DIGITAL CONTROL PERFORMANCE

The ultimate goal is always good control performance. Digital control systems can perform as well as equivalent continuous control systems under certain situations, but they have potential difficulties that should be considered in algorithm design. In this section, we apply the results of previous sections to evaluate digital control performance. We will begin with the standard PID feedback controller and proceed to IMC controllers.

PID CONTROL. The PID algorithm is easily implemented via either continuous (analog) or digital computation, with most new control equipment using digital. In general, better digital control performance results from faster execution, i.e., short execution periods. Guidance on PID performance is provided through the consideration of the following four examples.



EXAMPLE L.11.

Tune the PI controller controlling the heat exchanger and evaluate the dynamic response using simulation. For this example, we select the execution period as 2 minutes, although a commercial control system would typically execute the digital controller several times per second.

The dynamics for the process are known from previous examples. We will use the Ciancone PI disturbance correlation (Figure 9.9) modified as recommended in Chapter 11 for digital control, i.e., $\theta' = \theta + \Delta t/2$. The calculations are summarized below.

$K_p = -33.9$ K/(m³/min)	$\theta'/(\theta' + \tau) = 1/(1 + 11.9) = 0.078$
$\tau = 11.9\,\text{min}$	$K_p K_c = 1.3$
$\theta = 0\,\text{min}$	$T_I/(\theta' + \tau) = 0.23$
$\theta' = \theta + \Delta t/2 = 0 + 2/2 = 1.0\,\text{min}$	$K_c = 1.3/(-33.9) = -0.038$ (m³/min)/K
	$T_I = 0.23(12.9) = 3.0\,\text{min}$

The dynamic response for a step change in the set point is reported in Figure L.6 for the controller applied to the nonlinear heat exchanger model (given in Example 3.7). Note that the manipulated flow changes only every 2 minutes when the controller calculation is executed; between executions, the flow is maintained constant by the zero-order hold. The variables are well behaved, with the controlled variable returning to its set point and the controlled and manipulated variables experiencing smooth transitions, without undue oscillation or overshoot. Naturally, the performance of the digital control system with the long execution period is not as good as would be achieved by a continuous controller or a digital controller with short execution period.

By the way, the linearized, closed-loop system stability can be evaluated using equation (L.22). The roots of the characteristic equation are $0.76 \pm 0.27j$. Since the magnitudes of the poles are $0.80 < 1.0$, the poles are within the unit circle, and the system is stable. The poles for this tuning are farther from the unit circle than the poles for Example L.8. Thus, the Ciancone tuning in this example is more robust to model errors because it has a greater margin from the stability boundary.

> The digital PI controller can be tuned using modified tuning correlations to provide well-behaved dynamic responses; however, the control performance will degrade as the execution period is increased.

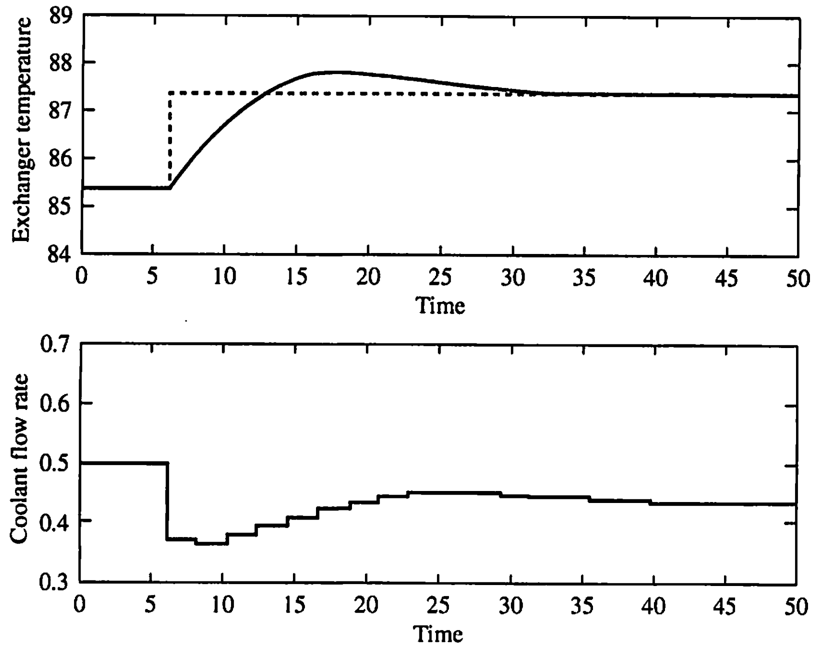
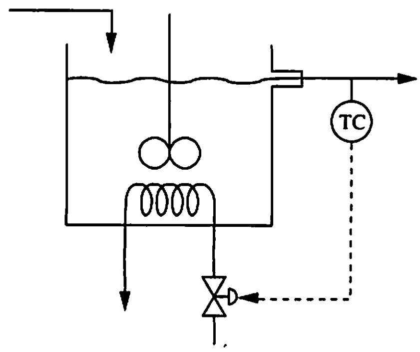


FIGURE L.6

Dynamic response for a set point change to a digital PI control of the stirred heat exchanger exit temperature determined in Example L.11.

EXAMPLE L.12.

Determine the final value of the heat exchanger control system for a step change in the set point of Δ SP.



The solution can be found (for stable tuning) by applying the final value theorem.

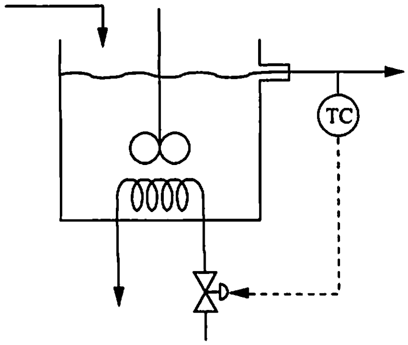
$$\lim_{z\to 1} \mathrm{CV}(z) = \lim_{z\to 1}(1 - z^{-1})\frac{\Delta \mathrm{SP}}{1 - z^{-1}} \frac{\dfrac{K_p K_c(1-B)z^{-1}\left(1 + \dfrac{\Delta t}{T_I}\dfrac{1}{1-z^{-1}}\right)}{1 - Bz^{-1}}}{1 + \dfrac{K_p K_c(1-B)z^{-1}\left(1 + \dfrac{\Delta t}{T_I}\dfrac{1}{1-z^{-1}}\right)}{1 - Bz^{-1}}}$$

$$\lim_{t\to\infty} \mathrm{CV}(t) = \Delta \mathrm{SP}\frac{K_p K_c(1-B)(\Delta t/T_I)}{K_p K_c(1-B)(\Delta t/T_I)} = \Delta \mathrm{SP}$$

Therefore, the digital control system with an integral mode returns the controlled variable to its set point, achieving *zero steady-state offset.*

Note that this important feature is achieved with a rectangular approximation to the integral calculation; a perfect, continuous integral is not required!

While the rectangular estimate is not exact, it provides a "persistent" adjustment of the controller output until the error returns to zero. Mathematically, this appears as a term $1/(1 - z^{-1})$ in the controller, which is required for zero offset at steady state.



EXAMPLE L.13.

Determine the performance of proportional-only feedback control for the stirred-tank heat exchanger. The controller execution period is increased to 15 minutes, which would occur when a sensor can provide a new measurement very infrequently. Use the value of the controller gain from Example L.11.

First, let us simulate the system and observe the performance. The dynamic response of the stirred heat exchanger with P-only control is given in Figure L.7. Both controlled and manipulated variables experience unacceptable oscillations. This poor performance might be unexpected, since this is the same value for the controller gain used in the PI controller, which gave acceptable performance in Figure L.6; only the integral mode has been removed.

Let us investigate the cause by determining the poles of the closed-loop system. The transfer functions for the process, equation (L.19), and the controller, $G_c(z) = K_c$, are substituted into the closed-loop transfer function, equation (L.21). Recall that $B = e^{-(\Delta t)/\tau}$ with $\Delta t = 15$.

$$\frac{\mathrm{CV}(z)}{\mathrm{SP}(z)} = \frac{\dfrac{K_p K_c (1-B)z^{-1}}{1-Bz^{-1}}}{1 + \dfrac{K_p K_c (1-B)z^{-1}}{1-Bz^{-1}}}$$

The denominator of this equation, the characteristic equation, is set equal to zero to evaluate the pole(s) of the system.

$$0 = (1 - Bz^{-1}) + K_p K_c (1-B) z^{-1} \tag{L.23}$$

Substituting the values for this example [$B = 0.284$, $K_p = -33.9$ K/(m³/min), and $K_c = -0.038$ (m³/min)/K]. we find the value of the pole to be $z = -0.64$. This pole is not close to instability, i.e., the boundary of the unit circle. However, this pole is located in the region near where the unit circle crosses the negative real axis

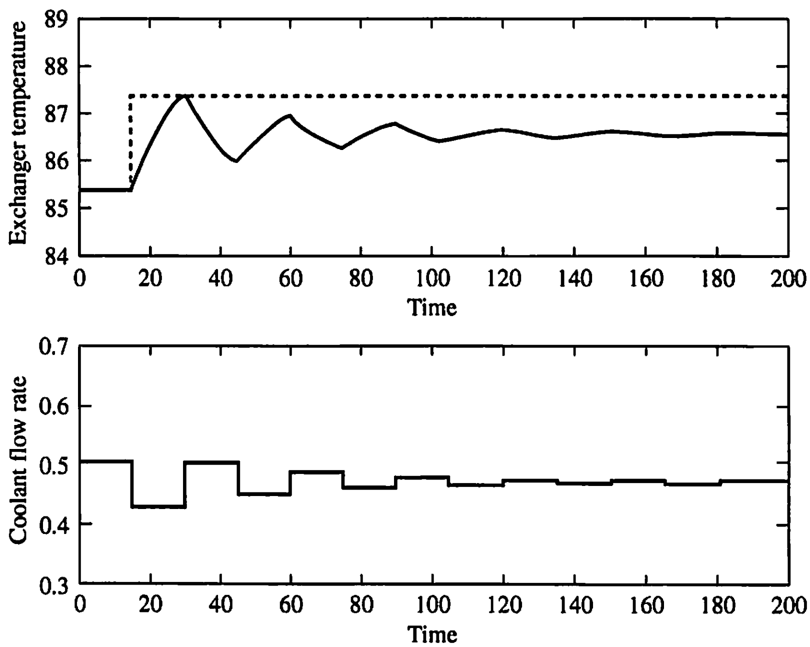


FIGURE L.7

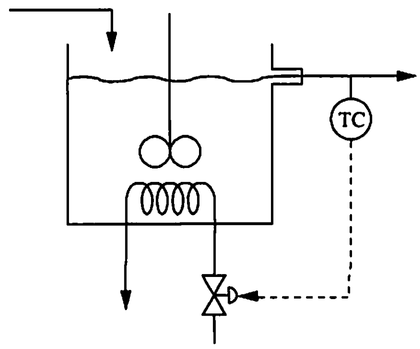
Dynamic response for digital P-only control of the stirred heat exchanger exit temperature determined in Example L.13. The poor performance is due to ringing.

$(-1, 0)$, which indicates that the sampled values will have a *ringing* behavior. This is exactly the behavior that we see in Figure L.7.

> Generally, a slowly sampled control system with proportional mode will tend to ring.

Tuning should be selected to reduce ringing while maintaining acceptable performance. One procedure is to lower the proportional gain and simultaneously decrease the integral time. When the sampling is so infrequent that the process essentially attains steady state between samples, the feedback controller should be changed to integral-only with $(K_c)/T_I = [(\Delta t)K_p]^{-1}$, which reduces the error to zero in one execution (if K_p is known exactly).

EXAMPLE L.14.



Determine the maximum controller gain that achieves stable behavior for a first-order process with proportional-only control and a 15 minute execution period.

We determine the answer to this question by using the characteristic equation of the closed-loop system, which was derived in the previous example as equation (L.23) and is repeated below.

$$0 = (1 - Bz^{-1}) + K_p K_c (1 - B) z^{-1}$$

We solve for the pole (z) to give

$$z = K_p K_c (1 - B) - B$$

We want to find the limiting value when $|z| = 1$. For negative feedback, $K_p K_c > 1$, and $B = e^{-(\Delta t)/\tau} > 0$. Therefore,

$$\text{Maximum } K_c = K_u = \frac{1}{K_p}\frac{1+B}{1-B} = \frac{1}{K_p}\frac{1+e^{-\Delta t/\tau}}{1-e^{-\Delta t/\tau}}$$

Recall that no stability limit to the controller gain exists for continuous, P-only control of a first-order system. The digital system is more restricted, but this result is consistent with our interpretation of the sample time as a type of dead time. Substituting the values for this example, the ultimate controller gain, $K_u = -0.0529$ $(m^3/min)/K$.

> A proportional-only feedback controller applied to a first-order process has an ultimate gain.

IMC CONTROL. The other major single-loop controller algorithm presented in this book is the IMC controller explained in Chapter 19. The IMC controller structure is repeated in Figure L.8. The following design criteria were determined for the IMC controller in Chapter 19 for an open-loop stable plant.

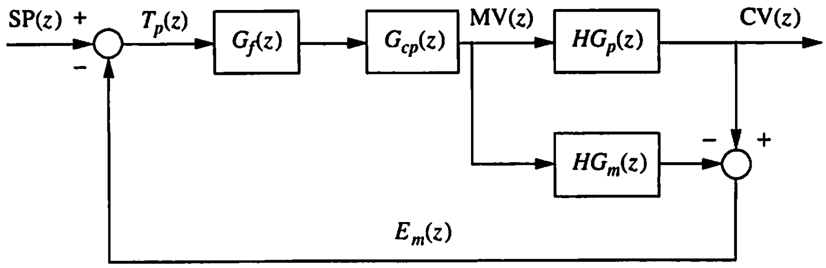


FIGURE L.8

Block diagram of digital IMC feedback control system.

1. The controller algorithm is an approximate inverse of the process model.

$$G_{cp}(z) \approx [HG_m(z)]^{-1}$$

The approximation is required to ensure that the controller is physically realizable. The design approach involves factoring the process model into invertible $[HG_m^-(z)]$ and noninvertible $[HG_m^+(z)]$.

$$HG_m(z) = HG_m^+(z)HG_m^-(z)$$

By convention, the steady-state gain of the noninvertible factor is selected to be 1.0. The controller is the inverse of the invertible factor.

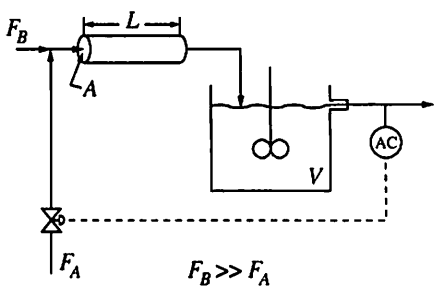
$$G_{cp}(z) = [HG_m^-(z)]^{-1}$$

2. The controller gain is the inverse of the process gain; $\lim_{z\to 1} G_{cp}(z) = \lim_{z\to 1}[HG_m^-(z)]^{-1}$
3. The controller must be stable.
4. A filter is included in the feedback path to modulate adjustments in the manipulated variable and to increase robustness to model error.

Here, the process transfer function includes dynamics of the final element (valve) and the sensor. We will design IMC controllers in the next three examples.

Here, the process transfer function includes dynamics of the final element (valve) and the sensor. We will design IMC controllers in the next three examples.

EXAMPLE L.15.



Design an IMC controller for the mixing process with dead time. Select the execution period (Δt) to be 1 minute.

The transfer function for this first-order-with-dead time process was derived in Example L.7 and is repeated in the following:

$$HG_p(z) = \frac{K_p(1 - e^{-\Delta t/\tau})z^{-1}}{1 - e^{-\Delta t/\tau}z^{-1}} z^{-5} = \frac{0.181z^{-6}}{1 - 0.819z^{-1}}$$

Let us try to design the controller as the exact inverse of the process model above.

Controller from inverse of $HG_m(z)$, transfer function:

$$G_{cp}(z) = \frac{\text{MV}(z)}{T_p(z)} = [HG_m(z)]^{-1} = \frac{1 - 0.819z^{-1}}{0.181z^{-6}}$$

For the moment, we will assume that no filter is included, which is satisfied with $G_f(s) = 1$. The digital controller would be implemented as a difference equation.

We apply equations (L.7) to convert the transfer function above to a difference equation, with sample "n" representing the current time.

Controller from inverse of $HG_m(z)$ difference equation:

$$\mathrm{MV}_n = 5.52(T_p)_{n+6} - 4.52(T_p)_{n+5}$$

The controller equation requires *future* values of T_p (the set point corrected by the model error feedback); therefore, this control calculation is not possible. As in the design of the continuous IMC controller, the approximate inverse must not invert the dead time. Factoring out the dead time and the *zero-order hold* (combined to give z^{-6}) from the numerator of $HG_m(z)$ and taking the inverse yields the following control algorithm:

Controller from inverse of $HG_M^-(z)$, transfer equation:

$$G_{cp}(z) = \frac{\mathrm{MV}(z)}{T_p(z)} = [HG_m^-(z)]^{-1} = \frac{1 - 0.819z^{-1}}{0.181}$$

Controller from inverse of $HG_M^-(z)$, difference equation:

$$\mathrm{MV}_n = 5.52(T_p)_n - 4.52(Y_p)_{n-1}$$

This calculation requires only *current and past* values and can be implemented; i.e., it is physically realizable.

Although the difference equation is realizable, the design can lead to a very aggressive feedback controller. The dynamic response for a set point change with a perfect model is given in Figure L.9. The variation in the manipulated variable is large and would not be acceptable for many chemical processes, e.g., manipulating distillation reboiler heating medium flow. Just as serious is the lack

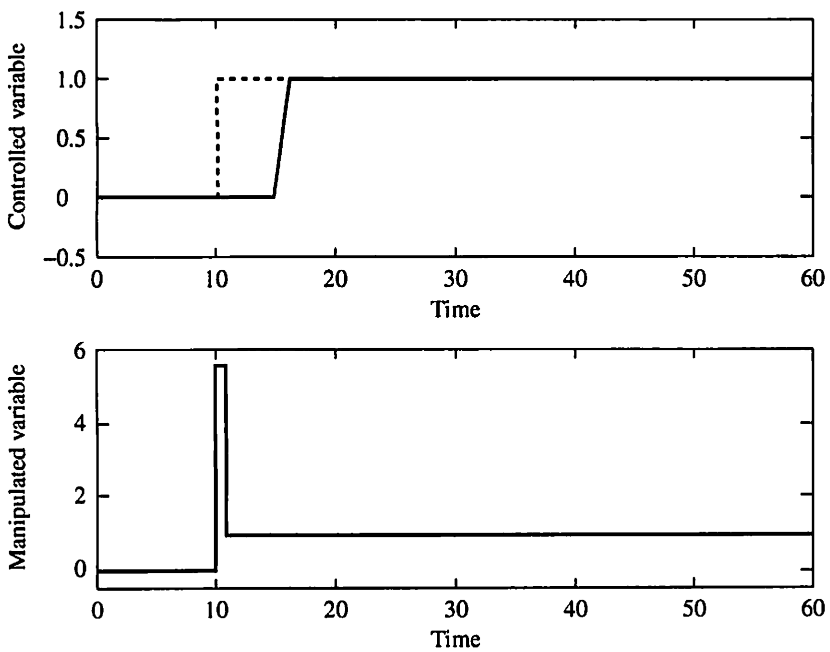


FIGURE L.9

Dynamic response for a set point change to a digital IMC controller without filter controlling the mixing process with dead time (no model mismatch). This controller is too aggressive for most applications. The results are from Example L.15 plotted in deviation variables.

of robustness; this controller would become unstable with modest modelling errors.

We can add a filter to the control loop to moderate the manipulation and increase robustness. Typically, one goal of the filter is to ensure that the power of z^{-1} in the denominator is at least as high as the power in the numerator. For this example, the natural choice is a first-order filter to give the following:

$$G_f(z)G_{cp}(z) = \frac{\text{MV}(z)}{T_p(z)} = G_f(z)[HG_m^-(z)]^{-1} = \frac{1-\alpha}{1-\alpha z^{-1}}\frac{1-0.819z^{-1}}{0.181}$$

We can use the tuning correlation for continuous IMC controllers provided in Figure 19.6 as a first estimate. The calculations are summarized in the following:

$$\theta = 5\,\text{min} \qquad \tau = 5\,\text{min} \qquad \theta/(\theta+\tau) = 5/10 = 0.50$$
$$\tau_p/(\theta+\tau) = 0.35 \text{ (from Figure 19.6)}$$
$$\tau_f = (0.35)10 = 3.5\,\text{min}$$
$$\alpha = e^{-(\Delta t/\tau f)} = e^{-(1/3.5)} = 0.75$$

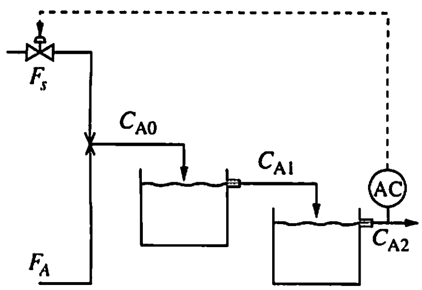
The controller in deviation variables using these variables is

Controller from inverse of $HG_m^-(z)$ and filter, difference equation

$$\text{MV}_n = 0.75\text{MV}_{n-1} + 1.38(T_p)_n - 1.13(T_p)_{n-1}$$

The closed-loop performance of the digital controller and filter is shown in Figure L.10*a* for a perfect model. While the approach of the controlled variable to its set point is slowed somewhat, the adjustment in the manipulated variable is more moderate and would be acceptable in most processes. The added robustness is demonstrated by the performance with model error (all plant parameters +25% from their estimated values) shown in Figure L.10*b*. Some degradation in performance is evident, but while not ideal, this performance would normally be acceptable.

> This example demonstrates that the IMC design procedures presented in Chapter 19 can lead to acceptable controller performance. Recall that the noninvertible factor includes the dead time *and the zero-order hold.*



EXAMPLE L.16.

Design an IMC controller for the series of chemical reactors in Example I.2. The concentration of the reactant leaving the second reactor (C_{A2}) is controlled by adjusting the solvent flow rate (F_s). Recall that this dynamic response experiences an inverse response due to two parallel paths: (1) the faster residence time effect and (2) the slower and stronger inlet concentration effect. The continuous, linearized model between the manipulated and controlled variables is derived in Example I.2. The transfer function model is second-order with numerator zero and is repeated in the following equation:

$$G_p(s) = \frac{C_{A2}(s)}{F_s(s)} = \frac{K_p(\tau_{\text{lead}}s+1)}{(\tau s+1)^2} = \frac{\dfrac{K_p\tau_{\text{lead}}}{\tau^2}(s+1/\tau_{\text{lead}})}{(s+1/\tau)^2}$$

The first step is to determine the z-transform of the process with a zero-order hold. The term $Z(G_p(s)/s)$ can be evaluated using entry 11 in Table L.1 (with $a_0 = 1/\tau_{\text{lead}}$

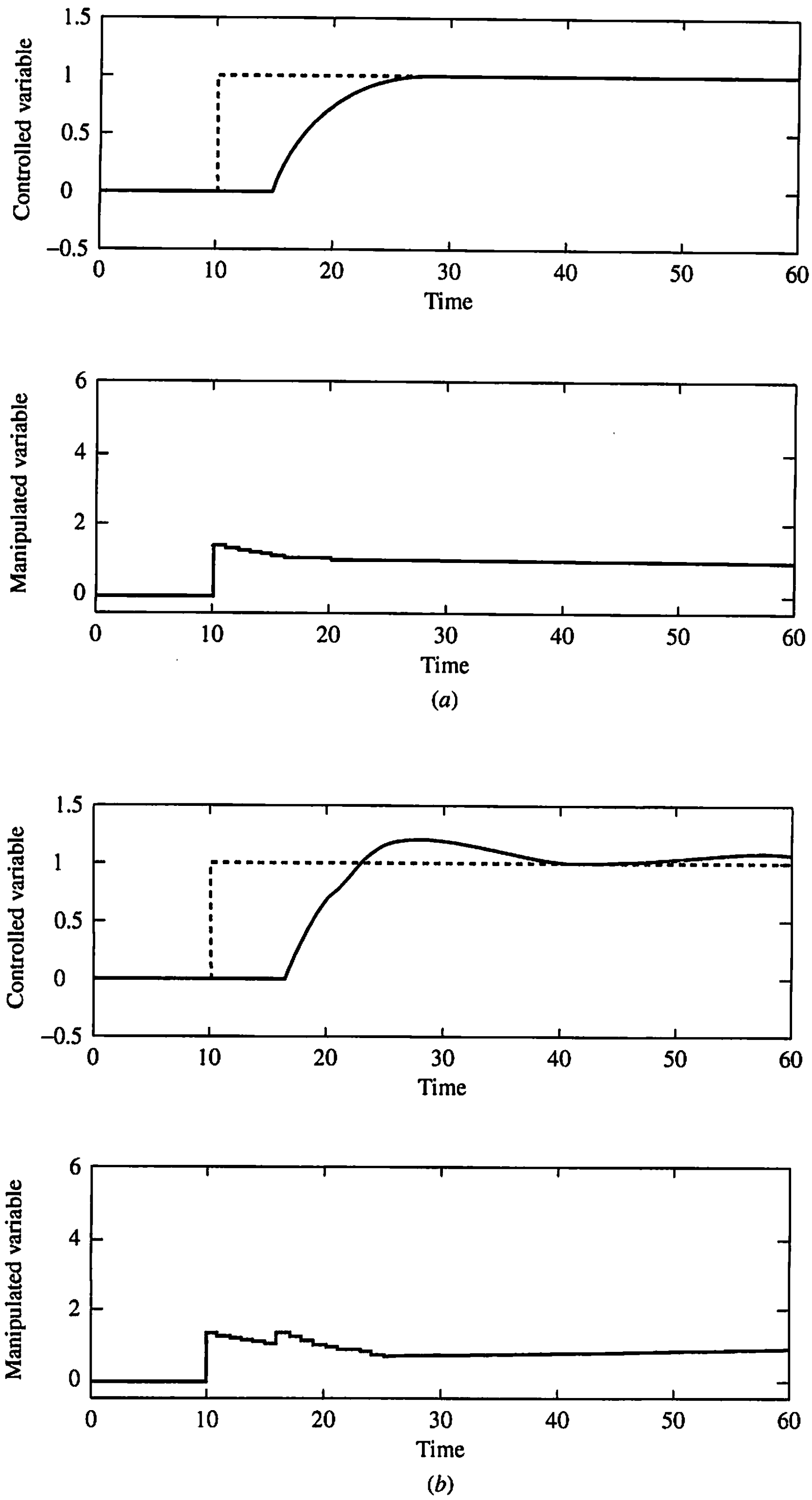


FIGURE L.10

Dynamic response for a set point change to a digital IMC controller with filter controlling the mixing process with dead time. The results are from Example L.15 plotted in deviation variables: (*a*) no model mismatch; (*b*) plant parameters 25% larger than the controller model.

and $a = 1/\tau$). The values for the parameters are

$$K_p = -1.66\ (\text{mole/m}^3)/(\text{m}^3/\text{min}) \qquad a = 1/\tau = 0.121\ \text{min}^{-1}$$
$$\tau = 8.25\ \text{min} \qquad a_0 = 1/\tau_{\text{lead}} = 0.125\ \text{min}^{-1}$$
$$\tau_{\text{lead}} = -8.0\ \text{min} \qquad \Delta t = 1.0\ \text{min}$$

When these values are substituted into the z-transform and terms with like powers of z^{-1} are combined, the following discrete process model with zero-order hold is determined:

$$HG_m(z) = \frac{C_{A2}(z)}{F_s(z)} = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

where $a_1 = -1.7717 \qquad b_1 = 0.1616$
$a_2 = 0.7847 \qquad b_2 = -0.1832$

Poles are 0.8858 (repeated) Zero is 1.1339

In the previous example, we found that the controller could be derived by inverting the process model without the dead time and zero-order hold; this process has no dead time and has a zero-order hold (z^{-1}). By taking the inverse of the model with the zero-order hold factored out, the following controller equation results:

$$HG_m^+(z) = z^{-1} \qquad HG_m^-(z) = \frac{C_{A2}(z)}{F_s(z)} = \frac{b_1 + b_2 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Controller from inverse $HG_m^-(z)$, transfer function:

$$G_{cp}(z) = \frac{F_s(z)}{T_p(z)} = \frac{1 + a_1 z^{-1} + a_2 z^{-2}}{b_1 + b_2 z^{-1}}$$

Recall that the input to the controller is the target, T_p, which is the set point minus the model error, $E_m = (C_{A2})_{\text{meas}} - (C_{A2})_{\text{pred}}$.

Before considering closed-loop performance, we evaluate the stability of the controller algorithm, $G_{cp}(z)$. We know that the stability of the transfer function depends on the poles of the transfer function. From the expression for the controller above, we see that the numerator zero in the process model becomes a pole in the controller. For this example,

> The controller pole $= -b_2/b_1 = 1.1339$, which has a magnitude greater than 1.0.

The controller is *unstable*, which is clearly not acceptable! Recall that the plant was stable, as indicated by poles of the process being located inside the unit circle. A zero outside the unit circle does not affect the plant stability, although it certainly affects the dynamic behavior, in this case giving an inverse response. To reiterate,

> The inverse of a stable plant can lead to an unstable controller, because the *zeros* of the plant are the *poles* of the controller.

To yield a stable controller, we must include in the $HG_m^+(z)$ all zeros that are outside the unit circle. If we were to simply factor out these zeros, we would change the gain of the remaining process model, $HG_m^-(z)$. Therefore, if we factor any zeros, we must compensate the gain of the remainder of the model. The procedure is demonstrated as we continue with the exercise.

Controller from inverse $HG_m^-(z)$ with the *unstable pole removed*, transfer function:

$$G_{cp}(z) = \frac{F_s(z)}{T_p(z)} = \frac{1 + a_1 z^{-1} + a_2 z^{-2}}{b_1 + b_2}$$

Note that when the unstable controller pole is removed, the contribution of the pole to the final value is retained, so that the controller gain is unchanged, as shown in the following expression:

$$\lim_{z \to 1}(b_1 + b_2 z^{-1}) = b_1 + b_2$$

Now, we have achieved a controller that is stable. Also the controller is causal because the current manipulated variable (F_s) depends on only current and past values. However, the controller would likely be too aggressive, so we want to add a filter that ensures that the orders of the numerator and denominator are equal. This will require a second-order filter, which we choose to be two first-order filters. The resulting controller is given in the following:

Controller from inverse $HG_m^-(z)$ with the unstable pole removed and a *filter added*, transfer function:

$$G_f(z)G_{cp}(z) = \frac{F_s(z)}{T_p(z)} = \frac{1 + a_1 z^{-1} + a_2 z^{-2}}{b_1 + b_2}\left(\frac{1-\alpha}{1-\alpha z^{-1}}\right)^2$$

We have designed a stable, causal controller that can provide robustness and moderate variation in the manipulated variable, with the proper choice of the filter time constant. No general studies are available to select the filter for this process model structure, so some trial-and-error tuning leads to the selection of 5 minutes for the filter time constant, to give $\alpha = e^{-(\Delta t)/\tau_f} = 0.819$, With this value, the controller algorithm becomes the following, which is clearly causal:

Controller from inverse $HG_m^-(z)$ with the unstable pole removed and *filter added*, difference equation:

$$(F_s)_n = 1.64(F_s)_{n-1} - 0.67(F_s)_{n-2} + 8.38(T_p)_n - 14.85(T_p)_{n-1} + 6.58(T_p)_{n-2}$$

where $\quad (T_p)_n = (SP)_n - [(C_{A2\text{meas}})_n - (C_{A2\text{pred}})_n]$

Recall that all variables in the difference equations are deviations from initial conditions. To calculate the actual flow, the initial condition of the solvent flow must be added to the value calculated above.

A sample set point response of the digital control system applied to the nonlinear series reactor process is given in Figure L.11. While the dynamic response is well behaved, no feedback controller can remove the poor performance resulting from the unfavorable process dynamics—specifically the inverse response. The performance achieved with the IMC controller is equivalent to the performance achieved with a proportional-integral controller, shown in Figure 13.15.

EXAMPLE L.17.

Reconsider the series reactors just analyzed in the previous example. Here, evaluate the effect of different sampling periods on the IMC controller design.

Naturally, the continuous process does not change when the sampling period changes; therefore, the continuous transfer function and the parameters K_p, τ, and

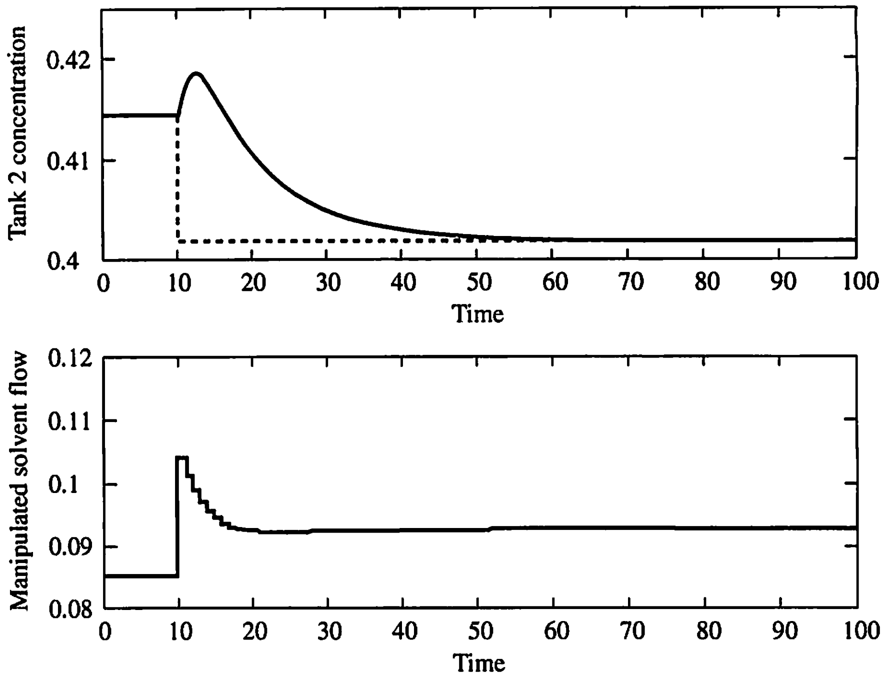


FIGURE L.11

Dynamic response for a set point change to a digital IMC controller with the unstable pole removed and with a filter. The process is a series of chemical reactors with an inverse response. The results are from Example L.16.

τ_{lead} do not change. Only the sample period (Δt) changes. However, because the sample period appears in most terms in entry 11 of Table L.1, all coefficients in the z-transform change. The general form of the z-transform is repeated in the following equation:

$$HG_m(z) = \frac{C_{A2}(z)}{F_s(z)} = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

We recall that the zero in the process model, $-b_2/b_1$ becomes the pole for the IMC controller. If this pole has a magnitude greater than 1.0, the controller will be unstable, which is not acceptable. The results below summarize the coefficients in the z-transform and the poles and zero for several values of the sample period.

Δt	a_1	a_2	b_1	b_2	Poles (repeated)	Zero
0.10	−1.9759	0.9760	0.0192	−0.1940	0.9880	1.013
1.0	−1.7717	0.7847	0.1616	−0.1832	0.8858	1.134
3.0	−1.390	0.4830	0.3205	−0.475	0.6950	1.480
10.0	−0.595	0.0885	0.0133	−0.8324	0.2976	63.63
15.0	−0.3246	0.0263	−0.4256	−0.7390	0.1623	−1.740
20.0	−0.1771	0.0078	−0.8110	−0.5680	0.0885	−0.7000

Magnitude less than 1

First, we see that the poles of the process always remain in the unit circle. The conclusion that the stability of an open-loop process does not depend on the sampling period certainly conforms to our expectation.

Second, we note that the magnitude of the zero becomes less than 1.0 for large sampling periods. We can understand this result by recognizing that the *sampled* values do not experience an inverse response if the period is sufficiently long; naturally, the inverse response occurs during intersample behavior. These results show that an IMC controller algorithm including the inverse of the zero would be *stable* for a period of 20 minutes.

However, we must evaluate the potential design further for a third important point. We note that the zero of the process becomes a pole of the controller, and a pole with a large negative real part will cause undesirable ringing. Therefore, this design with a pole at $(-0.7, 0)$ will not be acceptable. One method suggested by Morari and Zafiriou (1989) for achieving reasonable performance is to "remove" the ringing pole using the same method as used for removing the unstable pole, remembering to include the constant to maintain the controller gain unchanged.

> Digital IMC controller design must remove ringing poles to achieve acceptable performance.

We have applied IMC design to digital control in this section and have found that the design method, summarized in the four steps in the beginning of this section, is generally the same as for continuous systems. We have found one major difference; the design method must include a check for ringing pole in the controller. In addition, using z-transforms will enable us to apply the IMC design to more complex process models, not simply first-order-with-dead-time as in Chapter 19 on continuous processes. Finally, we have a direct manner of determining the difference equations for the controller and model calculations to be implemented in a digital computer.

L.5 □ CONCLUSIONS

In this appendix, we have developed a rigorous method for analyzing linear dynamic systems involving continuous processes and digital controllers. The z-transform of each component was derived and the individual transfer functions were combined using block diagram algebra to form an overall model. This model was applied to determine the stability, final value, and frequency response of digital systems.

Many of the results in this appendix were previously established through less rigorous studies in several chapters of the book. As we expect, delaying feedback control by increasing the execution period (1) requires tuning adjustments to maintain proper stability margins, and (2) degrades the control performance. In addition, we have learned about the new behavior of ringing, how ringing occurs in PID and IMC controllers, and how ringing can be avoided. Finally, we have learned how to derive difference equations for implementing digital calculations, and this method is easily implemented beyond the simple lead/lag described in Appendix F.

REFERENCES

Franklin, G., J. Powell, and M. Workman, *Digital Control of Dynamic Systems* (2nd ed.), Addison-Wesley, Reading, MA, 1990.

Morari, M., and E. Zafiriou, *Robust Process Control*, Prentice-Hall, Englewood Cliffs, NJ, 1989.

Smith, C., *Digital Computer Process Control*, Intext Educational Publishers, San Francisco, 1972.

ADDITIONAL RESOURCES

The following books provide extended analysis of digital control systems. The books by Jury have extensive tables of z-transforms.

Jury, E., *Sampled Data Control Systems*, Wiley, New York, 1958.

Jury, E., *Theory and Application of the z-Transform Method*, Wiley, New York, 1964.

Ogata, K., *Discrete-Time Control Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1987.

For more advanced coverage of IMC design, see Morari and Zafiriou (1989) above.